RELATIONSHIPS BETWEEN MATHEMATICAL PROOF AND DEFINITION

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This article discusses results from interviews with two undergraduate students in an introductory proofs course. The researcher assessed the participants' general proof schemes and built models of the participants' conception of probabilistic independence and mutual exclusivity. The participants were then tasked with asserting a relationship between independence and mutual exclusivity and trying to prove the asserted relationship. The results discuss possible interactions between students' conception of mathematical ideas and their approaches to proof.

Keywords: proof scheme, definition, concept image, probability

Mathematical proof and mathematical definition are two areas of research that have recently gained heightened attention from researchers (Edwards & Ward, 2004; Harel & Sowder, 1998; Ko, 2010; Vinner, 1991). These two areas, though generally studied separately, are intrinsically related (Edwards & Ward, 2004; Weber & Alcock, 2004). Little research, however, has explicitly explored relationships between proof and definition. The purpose of this research is to explore the connections and relationships between undergraduate students' proof schemes and their understanding and use of definition.

Theoretical Framework

A Framework for Discussing Proof

Harel and Sowder (1998, 2007) have provided a fundamental framework for research in students' conceptions of proof. This framework begins by defining proof as a two-part process: ascertaining and convincing. A student's notion "of what constitutes ascertaining and persuading" is called that student's proof scheme (Harel & Sowder, 1998, p. 244). There are varying levels of proof schemes described in the literature. Broadly, these proof schemes are External, Empirical, and Analytical — an External proof scheme includes validations by authority, ritual, or symbolism, an Empirical proof scheme is inductive or perceptual, and an Analytical proof scheme is more rigorous and logical. It is important to note that, at a given time, no person completely displays evidence of exactly one proof scheme. Because of this, a student's proof scheme is a generalization of the types of proof schemes evident through his or her work. For example, a student could exhibit both Empirical and Analytical proof schemes within a short period of time or even within a single proof; such a student could be said to have an "emerging Analytical" or "Empirical/Analytical" proof scheme.

Weber and Alcock (2004) also contribute a framework for discussing students' semantic and syntactic proof productions. This framework draws distinctions between students' use of instantiations in proof (syntactic) and the formal manipulation of logical mathematical statements (semantic). While this framework is constructed outside of Harel and Sowder's (1998) proof schemes, the two classification systems for students' mathematical tendencies seem as though they could work successfully together as neither excludes the other.

A Framework for Discussing Definition

Vinner (1991) distinguishes between the ideas of concept image and concept definition. He defines the concept image as a non-verbal entity such as a "visual representation of the concept… [or] a collection of impressions or experiences" (p. 68) which our mind associates with the concept. In contrast, the concept definition is the formal mathematical definition of a concept. These two ideas are not necessarily — and, one could argue, seldom — the same thing. Vinner uses the sentence, "my nice green car is parked in front of my house," as an example of concept image (p.67). He argues that the reader or listener does not necessarily consider the definition of each word in the sentence, but that each word invokes a generic concept image in his or her mind, the collection of which allows the sentence to take form as a whole impression.

Probability as a Context

Manage and Scariano (2010) provided a useful context in which this research was conducted. The authors found that most of the students in their study thought that two events being independent implied that they were mutually exclusive and vice versa. Although one's initial reaction may be to conclude this exact relationship, after careful consideration of the two concepts one realizes the two terms have almost exactly opposite meanings. This "almost" is attributed to cases in which one or both of the events has zero probability. Otherwise (i.e., if two events have nonzero probability), independence implies that two events are not mutually exclusive and mutually exclusive events are not independent. When asked to prove this relationship between independence and mutual exclusivity, one must address his or her own conceptions of independence and mutual exclusivity, compare the two concepts, ascertain the relationship between the two, and try to convince others. So, this relationship between mutual exclusivity and independence will provide a context for exploring the use of definition in proof.

Methods

The researcher conducted semi-structured interviews with three undergraduate students – Alex, Betty, and Caroline – who were enrolled in an Introduction to Proofs course. All three students were mathematics majors in their third year and were chosen randomly from a group of volunteers. None of the students were compensated for their participation. Each participant completed three interviews, each lasting approximately one hour.

Each of the three interviews had its own unique goal: (Interview 1) to gauge the participants' general proof schemes, (Interview 2) to gain insight into the participants' concept definitions and concept images of specific mathematical terms, and (Interview 3) to observe and analyze the participants' use of definition and imagery while proving relationships about the mathematical terms discussed in the previous interview.

Data Collection

All interviews were recorded using both video and audio devices. The researcher kept notes throughout the interviews and all participant work was collected. The first interview was designed to gather a general understanding of each participant's proof scheme. The interview consisted of each participant assessing a "matrix of proofs," which is a 3-by-3 grid of mathematical proofs. Each row in the matrix contained three variations of proof of the same mathematical relationship, reflecting Harel and Sowder's (1998) three major proof schemes-Analytical, External, and Empirical. The participants were asked to assess each proof for mathematical and logical correctness. From these responses, the researcher determined the aspects of mathematical proof that the participants considered important and/or necessary or, conversely, unimportant and/or unnecessary. In turn, the researcher used the participants' responses and reasoning in order to form a notion of each participant's general proof schemes.

In the second interview, the researcher collected the participants' definitions of mutually exclusive events and independence. The researcher also asked the participants to consider several events in various sample spaces and determine whether pairs of events were mutually exclusive and/or independent. Participants were also invited to introduce their own events and sample spaces to elaborate points that came up during discussion. This was intended to provide the interviewer with insight not only into how the participants defined each of the two terms, but also how these terms were applied in various probabilistic contexts. The researcher could then distinguish between the participants' concept definitions (collected directly) and concept images (drawn from examples, phrasing, etc.).

The third interview was designed to provide a context wherein the participants could assert a mathematical relationship between independence and mutual exclusivity and then attempt to prove this relationship. The participants were asked to assert two main relationships: given that two events have nonzero probabilities in the same sample space, does independence imply mutual exclusivity and does mutual exclusivity imply independence? These questions were posed as two separate multiple-choice questions, as in Manage and Scariano (2010).

Data Analysis

The researcher analyzed each video and audio recording after each interview in order to determine the participants' proof schemes, identify major themes in the participants' reasoning, model participants' understanding of mutual exclusivity and independence, draw quotes from the dialogue to support such models, and develop individualized clarifying tasks for the subsequent interview. Throughout the video analysis, video clips were taken that supported or challenged working models of participant thinking. These videos were collectively re-analyzed in order to confirm or reevaluate a model. The researcher would then develop questions for the subsequent interview that would be used to help clarify conflicts within the model.

Results

For the sake of depth and limited length, we discuss only Alex and Betty here.

Alex

Throughout the first interview, Alex exhibited a predominately Analytic proof scheme. Eventually he correctly supported all Deductive proofs and refuted all Empirical and External proofs, citing appropriate flaws in logic or reasoning. In a few instances, he showed signs of relying on a proof's form rather than content, signifying an occasional tendency toward a Ritual (External) proof scheme. Alex was also very pedantic about precise details, reflecting a skeptical point of view and checking for logical progression in each proof.

Alex displayed a deep understanding of examples and their use in proof. This was quickly evident in the first example in the matrix of proofs. This proof, applying an Inductive (Empirical) proof scheme, used an example of a large random number that exhibited the desired result. After reading the argument, Alex immediately said, "Yeah, this is bogus." He later refuted other inductive proofs very similar to the first. These examples highlight Alex's ability to refute proofs that inappropriately use examples.

Another interesting aspect of Alex's proof scheme is his emphasis on the axioms of real numbers and considering the space in which he was working. These qualities were evident in three instances. In the first two cases, Alex explicitly applied the axiom of the closure of integers under addition and multiplication. In the second case, Alex also invoked the associativity axiom for real numbers. In the third case, Alex suggested that the sum of the interior angles of a triangle might not be 180° in non-Euclidean space. While this could be a manifestation of the rigor required in his Introduction to Proofs course, it is evident from these examples that Alex had internalized a mindset that considers the system in which a proof is argued and its fundamental axioms. It should be noted that Alex's use of axioms in this interview reflects Harel and Sowder's (1998) Intuitive-Axiomatic proof scheme.

These qualities of Alex's proof scheme combine to support an initial emphasis on the form of a proof and then careful investigation of motivation and justification at each line of an argument. This emphasis was manifested in his pedantic discussions of the proof writer's justification, his use of axioms, and refutation of proofs by example. We see Alex used the form of a proof to make initial judgments, but his skepticism forced him to evaluate a proof based on its line-by-line merit. From this, we can conclude that Alex generally displays an Intuitive-Axiomatic (Analytical) proof scheme with tendencies toward a Ritual (External) proof scheme.

In the second interview Alex explored two sample spaces and discussed a few other examples that he used to help describe his understanding of mutual exclusivity and independence. As we will see, Alex displayed an extremely internalized and powerful conception of independence. Alex defined independence as, "[when] the outcome of one event does not affect the outcome of a subsequent event." This definition implies an emphasis on a sequence of events, where one of the events being considered must occur prior to the other.

With regard to mutual exclusivity, however, Alex was less certain of a formal definition - changing his definition twice throughout the interview until eventually declaring, "[Mutual exclusivity is when] performing an event or series of events causes a subsequent event to have zero probability of happening." Again, Alex uses the word "subsequent" in his definition, which implies that this relationship is defined over a period of time. It is important to note that Alex's initial definition of mutual exclusivity (consistent with the mathematical definition) was not defined over time, but rather instantaneously. It was not until he had considered examples in the two given sample spaces that he changed this definition to more closely resemble his definition of independence.

When prompted for an example of independent events, Alex gave two examples: a die and a coin. He stated that rolling a six on the first roll of a die does not affect rolling a six on the second roll of a die and gave an analogous explanation for the coin. These examples are consistent with his definition of independence, implying that the two events in consideration take place at separate times. His initial examples of mutually exclusive events exhibited what he described as "well-defined states" including raining versus not raining, sides of a die ("you can't roll both a 5 and a 1"), and a coin ("it's either heads or tails"). These examples support his original definition, which considers the two outcomes instantaneously in that it cannot both rain and not rain at the same time. Later in the interview, however (after changing his definition of mutual exclusivity), Alex described repeatedly drawing "any card" without replacement until all spades were exhausted. In this case, drawing "any card" and drawing a spade were mutually exclusive since drawing "any card" can eventually cause drawing a spade to have probability zero. This example seems much more convoluted than the first three examples, but supports Alex's newer definition of mutual exclusivity.

We can see that Alex's conception of independence was so strong that it not only influenced how he defined mutual exclusivity, but also caused him to reject three different examples and develop a new concept image for mutual exclusivity wherein one event must cause a subsequent event to be impossible. This new concept image was so strong that, when asked to reconcile this new definition with his original examples, Alex reneged on their mutual exclusivity (e.g., heads on a coin does not cause "not tails" later).

Equally intriguing is the fact that Alex independently asserted a corollary to his new definition of mutual exclusivity. In this corollary, Alex stated that if the two events are mutually exclusive, then they cannot be independent. This reflects the (almost) exact relationship outlined in Manage and Scariano (2010) and investigated in the third interview of this research. Alex used an explanation analogous to that described in Manage and Scariano (2010). He asserted that, since one event causes the second event to have zero probability, the first event changes the probability of the second event and therefore the two events are not independent. It should be noted, however, that Alex did not consider the case when the second event in the sequence already had zero probability.

In the third interview, Alex responded that if two events were mutually exclusive this implied that they were not independent. This claim was made using his final definition of mutual exclusivity. He directly referenced his own corollary from the second interview in which he made this exact assertion. Alex also claimed that if two events are independent then they are not mutually exclusive. He supports his answer choice by saying, “one event’s not affecting the other event at all so, I mean, it’s not going to cause it to have zero probability cause it’s not changing the probability of the next event.” As with the first question, this answer choice supports the relationship between the mathematical definitions of independence and mutual exclusivity for nonzero events.

Betty

Betty displayed a predominately Analytic proof scheme with the exception that she accepted one proof based on its appearance and another proof based on its form. Betty correctly refuted the three examples of Inductive (Empirical) proofs, but accepted one Deductive proof because it “seem[ed] more mathematical.” Her refutation of the inductive proofs shows her understanding of the importance of a general proof for all cases. Betty’s acceptance of a proof based on its seeming mathematical qualities and acceptance of a false proof by the principle of mathematical induction, however, indicate a tendency toward External (Ritual) and Empirical (Perceptual) proof schemes.

Betty showed an insistence on understanding very specific aspects of a proof rather than drawing any assumptions about the proof’s process with the exception of one case. She quickly accepted a proof by mathematical induction. Here, she was likely preoccupied with the form (or “look”) of the proof, rather than its mathematical validity. This idea was supported when Betty stated that her class had recently discussed the principle of mathematical induction. When asked which of the first three proofs she preferred, Betty chose the last proof because the processes in the second proof were not obvious to her. This reflects a need to understand connections in a proof, even though this need was temporarily suspended in the case of mathematical induction. This need was also addressed later in the interview, when Betty described the process of verifying for herself relationships she felt she did not understand in class.

Throughout the rest of the first interview, Betty correctly refuted Empirical and External proofs and accepted Analytical proofs. She rejected the false proofs with little hesitance. At one point, Betty described combinations of negating the hypotheses statements of the Inductive proofs, showing a clear understanding of logical proof, counterexamples, and proof by contradiction. She also reflected an ability to identify false proof by example. These examples show a healthy skepticism of Authoritarian and Ritual proof, both of which are External proof schemes. Additionally, Betty’s explanations in refuting Inductive proof schemes support an emphasis on proof for all cases.

When asked what it meant for two events in a sample space to be independent, Betty responded, “The intersection is zero. Is it? That’s what I’m asking. I don’t remember.” Betty almost instantly changed this to, “Two events are independent if the probability of A occurring does not affect the probability of B occurring.” Betty then described the independence of events A and B using the equation $P(A) = P(A|B)$. Neither of these representations necessarily implies a chronological distinction between events A and B (as was seen with Alex's use of the word "subsequent"). But, when prompted for an example of independent events, Betty described the act of picking a card from a deck of fifty-two cards and putting it back so that the probability of picking a second card is not affected. Similarly, when asked for an example of events not being independent, Betty provided the case of picking a card and not replacing it. These examples are consistent with a conception of independence in the context of a "with replacement" and "without replacement" conditioning event.

In contrast, Betty defined mutually exclusive events with the statement, "you can't have both at the same time." This definition explicitly states that the events can be compared instantaneously. Here, Betty gave the example that the choosing the queen of hearts and choosing the jack of diamonds are mutually exclusive because they cannot both occur when one card is drawn. We notice that this definition is consistent with the mathematical definition and that this example is consistent with Betty's definition. Betty did spend much more time defining mutually exclusive events compared to her definition of independence, but once she determined this definition, she held firm to its accuracy saying, "I'm sorted now." This reflects her need in first interview to prove relationships in order to understand them.

Betty's initial confusion of independent events as events that "don't happen at the same time" reflects the most common misconception in Manage and Scariano (2010). Although she quickly changed her mind about the definition of independence, this confusion was apparent in her use of mathematical notation to represent the ideas (discussed below). Also, when explaining her conditional notation of independence, Betty described two independent events as "completely separate," which one could argue is a descriptor more applicable to mutually exclusive events since their intersection is empty.

More than once, Betty wrote an equation involving probabilities saying, "That's just something I remember from probability." For instance, she initially used "$P(A \cap B)=0$" to represent independence and used the equation "$P(A \cap B)=P(A)*P(B)$" to define mutual exclusivity. These equations were quickly erased. The former, however, was eventually used to describe mutual exclusivity. For the latter, Betty admitted, "[I have] no idea where that came from or if that's even mutually exclusive. And I would not be able to come up with [it]." We notice that Betty's second description of independence, $P(A)=P(A|B)$, is true unless the probability of B is zero. In this case, the statement $P(A|B)$ makes no sense, although it could be adapted to say, "two events A and B are independent if both have nonzero probability, $P(A)=P(A|B)$, and $P(B)=P(B|A)$."

In this interview, we see that Betty's concept definitions, though initially inconsistent, are each strongly internalized when evaluating the independence and mutual exclusivity of specific events in specific sample spaces. This is evident because once Betty defined each term, she was "sorted" on how to verify them and seemed to develop quick checking schema in order to do this (e.g., "Can these happen at the same time?"). Her spoken reasoning for two events' independence and mutual exclusivity reflected these quick checks.

In response to each of the two questions in the third interview, Betty concluded that there was not enough information about the sample space and that two mutually exclusive events can be both independent and not independent. This led her to respond that there was not enough information about the sample space or the context of selecting events in the sample space to determine a relationship. She explained that in the previous interview she had seen mutually exclusive events that were both independent and not independent (a copy of her responses from the second interview was presented to her). She also explained that she saw independent events that were both mutually exclusive and not mutually exclusive in the second sample space. From this, Betty reasoned that more information was needed about both the sample space and the actions taken between the occurrence of the first event and second (e.g., replacement, non-replacement). Again, we see independence is affected by the context in which the events take place.

Betty's proof scheme showed that she is more inclined to want to verify mathematical relationships on her own. This was evident as she "sorted" herself about the definitions of independence and mutual exclusivity. During this process, Betty successfully reconciled her definitions of the terms with symbolic representations (about which she was admittedly unsure) that she had recalled from her statistics course. Betty used these definitions to investigate the sample spaces in the second interview, the results of which had a direct affect on her reasoning in the third interview. Because Betty's definition of independence relied so heavily on the sample space and whether replacement occurred, she had examples of all different combinations of independence and mutual exclusivity.

Conclusion

We see that proof schemes can be both restricted and enhanced by students' definitions of the mathematical ideas they consider. Though her reasoning was logically based on her previous experiences in the samples spaces, Betty's conception of independence and mutual exclusivity caused her to require more information about the sample spaces in question, in turn restricting her ability to draw conclusions between the two concepts. On the other hand, Alex's ability to adapt his concept image and concept definition of mutual exclusivity allowed him to logically conclude both directions of the relationship between mutual exclusivity and independence, however correct or incorrect his definition may have been.

In his proof, Alex claimed from his concept definition of mutual exclusivity that each mutually exclusive event would cause the other to have zero probability. This would make the two events "not independent" since his definition of independence necessitated each event to "not affect a subsequent event." Using similar reasoning, Alex concluded that independence implied "not mutual exclusivity." It should be noted however that, despite Alex's focus on "proof for every case" in the first interview, he failed to assert a relationship for the case when one or both events were given to have zero probabilities. The contrast between his assertions about proof and his actions in proving this relationship reflects the "pathological" nature of zero probability cases pointed out by Kelly and Zwiers (1986). Interestingly, this also points to a characteristic of his definitions that may have influenced his thought process: an event with zero probability cannot "happen first" and therefore can neither cause nor affect any other event, as the definitions require.

Betty's was unable to logically assert any certain relationship between the two concepts. This resulted from such strong concept images of independence and mutual exclusivity. More specifically, Betty's personal experiences in the sample spaces allowed her to provide counterexamples to any explicit relationship between the two concepts. Since specific characteristics of sample spaces affected two events' independence, she required information about a sample space in order to make inferences about the events in question. This prevented Betty from generalizing to all cases an explicit relationship between mutual exclusivity and independence, which her proof scheme required.

Recalling the Alex and Betty's general proof schemes (mostly Analytical with slight Empirical and External tendencies), we consider how these related to their use of definition. Alex's dynamic concept image and unsolicited production of the lemma for the definition of mutual exclusivity reflect an Analytical frame of mind that is also geared toward finding and asserting relationships between the two concepts. We see with Betty, however, that a mostly Analytical proof scheme alone is not sufficient to connect the relationships between mutual exclusivity and independence. This is because her conceptions of the two ideas were so powerful that she was comfortable using the four cases from her exploration to show that no relationship existed. From these two cases, we see that little inference can be made about how a student uses definition relative to Harel and Sowder's (1998) proof schemes.

But we can also consider these cases with respect to Weber and Alcock's (2004) semantic and syntactic proof productions. Because he produced it immediately after changing his concept definition of mutual exclusivity to more closely resemble his concept definition of independence, we see that Alex's lemma (and therefore responses in the third interview) was a direct result of his comparing the two concept definitions. A syntactic approach to the relationship was not fruitful, however, until he changed his definition. Conversely, Betty's use of previous instantiations (a semantic approach) prevented a definite relationship between the concepts from forming. It is unclear, though, whether Betty even thought her concept definitions might need to be changed. From this, we see some indication that a syntactic approach may play some role in aiding the adaptability of definition and that a semantic approach could be more restrictive.

From this research, we have seen how the adaptability of a student's concept image allows him or her to compare seemingly disparate concepts in new contexts. Here, the phrase "seemingly disparate" reflects the understanding of the concepts from the students' initial points of view. This action reflects Vinner's "interplay between definition and image," but is different in that the participants were not comparing a definition and image of a single mathematical concept, but rather two different but related images (1991, p. 70). This interplay is not addressed in his work, but yields a result similar to that of Vinner's interplay where an adaptation of image allows one to make sense of a perceived relationship. In this case, the adaptation of two images allowed a relationship to be perceived. Conversely, in Betty's case, rigidity restricted her perception of a relationship between independence and mutual exclusivity.

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