Re-claiming: One way in which conceptual understanding informs proving activity

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Abstract: In this research, I set out to elucidate the construct of Re-Claiming - a way in which students' conceptual understanding informs their proof activity. This construct emerged during a broader research project in which I analyzed data from individual interviews with three students from a junior-level Modern Algebra course in order to model the students' understanding of inverse and identity, model their proof activity, and explore connections between the two models. Each stage of analysis consisted of iterative coding, drawing on grounded theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). In order to model conceptual understanding, I draw on the form/function framework (Saxe, et al., 1998). I analyze proof activity using Aberdein's (2006a, 2006b) extension of Toulmin's (1969) model of argumentation. Reflection across these two analyses contributed to the development of the construct of Re-Claiming, which I describe and explore in this article.

Key words: Mathematical Proof, Conceptual Understanding, Abstract Algebra

Mathematical proof is an important area of mathematics education research that has gained emphasis over recent decades. Inherent in the process of proving is the notion that one must validate (or refute) some mathematical relationship that one might not necessarily know before he or she engages in the activity of proving. Each proof involves the statement of a mathematical relationship, which is either intuitively driven or presented to the individual, and the validity of which is either in question or taken as unknown. The individual then sets out to draw on his or her specific notions about the concepts involved in the relationship in order to show that the relationship is valid relative to his or her own mathematical reality, logic, reasoning, and perception of expectations within a mathematical community in which he or she might intend to communicate such proof activity. Once the relationship is validated (or refuted), there is new potential for the prover to begin to incorporate this new relationship into his or her understanding of the concepts involved (perhaps slowly and over time, perhaps quickly and with immediate consequences). In this brief description of the proving process, one might identify two general interactions: the ways in which a prover's current conceptual understanding informs proving activity in the moment and the potential the individual has to alter his or her understanding of the very concepts about which he or she is proving.

The majority of empirical research in proof focuses on individuals' proof production (e.g., Alcock & Inglis, 2008), individuals' understanding of or beliefs about proof (e.g., Harel & Sowder, 1998), and how students develop notions of proof as they progress through higher-level mathematics courses (e.g., Tall & Mejia-Ramos, 2012). Researchers have also generated philosophical discussions that explore the purposes of proof (e.g., Bell 1976; de Villiers, 1990). Much of this latter discussion centers on the explanatory power of proof (e.g., Weber, 2010), with the primary focus being on the techniques and methods involved in a given proof (e.g., Thurston, 1996), rather than the development of concepts or definitions (Lakatos, 1976). Few studies, however, use grounded empirical data to explicitly discuss the relationships between an individual's conceptual understanding and his or her engagement in proof (e.g., Weber, 2005). Rather, research tends to isolate proof as a discipline in-and-of itself – relatively decontextualized from the specific mathematical conceptions the prover brings to bear in a given situation. In this research, I focus on individual students' engagement in Abstract Algebra proofs that involve inverse and identity. Specifically, I seek to investigate the question: "*How might students' conceptual understanding of identity and* *inverse relate to their proof activity?*" In this article, I introduce a construct, re-claiming, that addresses one way in which conceptual understanding informs proving activity. I then provide examples from one participant's interview responses that illustrate the underlying interactions between conceptual understanding and proof activity involved in re-claiming.

Theoretical Frameworks

In this research I operationalize participants' conceptual understanding using Saxe et al.'s constructs of form and function (Saxe, Dawson, Fall, & Howard, 1996; Saxe & Esmonde 2005; Saxe et al, 2009). Throughout the literature, forms are defined as cultural representations, gestures, and symbols that are adopted by an individual in order to serve a specific function in goal-directed activity (Saxe & Esmonde, 2005). Three facets constitute a form: a representational vehicle, a representational object, and a correspondence between the representational vehicle and representational object (Saxe & Esmonde, 2005). Saxe focuses on the use of forms to serve specific functions in goal-directed activity as well as shifts in form/function relations and their dynamic connections to goal formation. Through this framework, learning is associated with individuals' adoption of new forms to serve functions in goal-directed activity as well as the development of new goals in social interaction. Saxe, Dawson, Fall, and Howard (1996) describe how one might think of learning using *form/function* relations, saying, "Mathematical development in the form/function framework can be understood as a process of appropriating forms that have been specialized to serve developmentally prior cognitive functions and respecializing them such that they take on new properties" (p. 126). Accordingly, the *form/function* framework provides an appropriate theoretical framing for investigating the ways that individuals' understanding of identity and inverse relates to their engagement in the goal-directed activity of proving.

Saxe (1999) discusses how a *form* can be schematized as a vehicle for mathematical meaning. This schematization involves a representational vehicle, a representational object, and a semantic mapping (correspondence) between vehicle and object (p. 23). Saxe (1999) goes on to state that, "inherent in the microgenetic act is a schematization of a correspondence between the latent qualities of the vehicle and object such that one can come to stand for the other" (p. 24). He continues, "individuals structure cultural forms … into means for accomplishing representational and strategic goals. This dynamic process allows for the flexibility of forms to serve different functions in activity, in that the same forms may be structured into means for accomplishing different ends" (Saxe, 1999, p. 26). These quotes draw focus toward the ways in which *forms* are able to shift during goal-oriented activity. This aspect of the *form/function* framework informs the focus of the current study by drawing attention to the ways in which participants structure the *forms* and *functions* upon which they draw during proof activity, specifically with regards to the ways in which specific forms might support varied reasoning within different proving contexts.

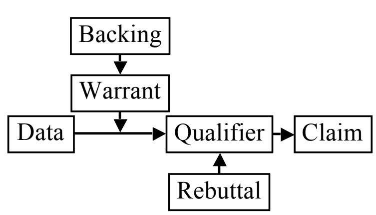

Figure 1. Visual representation of Toulmin models

In order to model participants' proof activity, I use Aberdein's (2006a) adaptation of Toulmin's (1969) model of argumentation. Several researchers have adopted Toulmin's model of argumentation to document proof (e.g., Fukawa-Connelly, 2013). This analytical tool organizes arguments based on the general structure of claim, warrant, and backing. In this structure, the claim is the general statement about which the individual argues. Data are general information, facts, rules or principles that support the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use backings, which support the warrant; rebuttals, which account for exceptions to the claim; and qualifiers, which state the resulting force of the argument (Aberdein, 2006a). This structure is typically organized into a diagram, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

Aberdein (2006a) provides a thorough discussion of how Toulmin models might be extended to organize mathematical proofs, including several examples relating the logical structure of an argument to a Toulmin model organizing it. Using "layout" to refer to the graphic organization of a Toulmin model, Aberdein includes a set of rules he to coordinate more complicated mathematical arguments in a process he calls combining layouts: "(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout" (p. 213). The first two rules are relatively straightforward – the first focuses on the treatment of the graphical layout, as for the second, one can imagine including multiple data sources in the same data node. The third and fourth rules provide a structure for combining different layouts and rely on organizational principles that Aberdein uses. He provides examples of combined layouts (Figure 2).

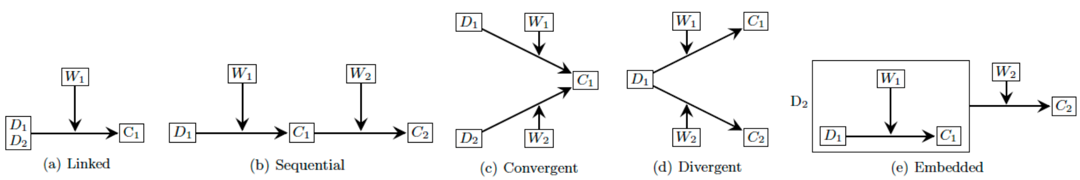

Figure 2. Five Ways of Combining Layouts (Aberdein, 2006a, p.214)

Together, these two theoretical frameworks support the development deeper models of the participants' conceptual understanding and proof activity. Further, juxtaposition of these models, affords more holistic insight into the motivations of participants' proof activity, specifically contextualizing the argumentation that a participant develops in the moment relative to documented consistencies in his or her ways of thinking as well as affording insight into subtle, in-the-moment shifts in the ways that participants draw on and use the very concepts about which they are proving. This directly addresses a student's proof activity as situated relative to his or her conceptual understanding, each of which is treated as an emergent, dynamic facet of the student's mathematical reality. In short, the combined use of these frameworks allows researchers hold both concept and activity as integral and integrated, each continually informing the other throughout the student's proving process.

Methods

Data were collected with nine students in a Junior-level introductory Abstract Algebra course, entitled *Modern Algebra.* The course met twice a week, for seventy-five minutes per meeting, over fifteen weeks. The curriculum used in the course was *Teaching Abstract Algebra for Understanding* (TAAFU; Larsen, 2013), an inquiry-oriented, RME-based curriculum that relies on Local Instructional Theories that anticipate students' development of conceptual understanding of ideas in group theory. Three individual interviews (forty-five to ninety minutes each) took place at the beginning, middle, and end of the semester, respectively. These interviews were semi-structured (Bernard, 1988) and used a common interview protocol so that each participant was asked the same questions as the others. Unplanned follow-up questions were asked during the interview to probe students' descriptions and assertions. The goal for each interview was to evoke the participants' discussion of inverse and identity and engage them in proof activity that involved inverse and identity. I developed initial protocols for these interviews, which were then discussed and refined with fellow mathematics education researchers.

Each interview began by prompting the student to both generally describe what "inverse" and "identity" meant to them and also to formally define the two mathematical concepts. Additional follow-up questions elicited specific details about what the participant meant by his/her given statements, figures, etc. The interview protocol then engaged each participant in specific mathematical tasks aimed to elicit engagement in proof or proof related activity. Each participant was asked to prove given statements, conjecture about mathematical relationships, and describe how he or she might prove a given statement. As with the questions about defining, each of these tasks had planned and unplanned follow-up questions so that all participants were asked at least the same base questions, but their reasoning was thoroughly explored. Throughout the interviews I kept field notes documenting participants' responses to each interview task. I also audio and video recorded each of the interviews, and all participant work and the field notes were retained and scanned into a PDF format. As I shifted to analysis of the data, I narrowed my focus to three of the participants (John, Tucker, and Violet) and transcribed each interview, including thick descriptions of participants' gestures and the timing of pauses in participants' speech.

The retrospective analysis of the three participants' interview responses consisted of three stages, which I ordered so that each stage built upon the previous stages toward a resolution of the general research question. This consisted of an iterative coding process to generate thorough models of the participants' conceptual understanding and engagement in proof and proof-related activity. I analyzed each participant's data separately, coordinating each analysis chronologically so that the model of each participant's conceptual understanding corresponded with his or her proof activity over the semester. I then investigated relationships between each participant's conceptual understanding and proof activity, exploring instances in which meaningful interactions between understanding and activity occurred.

Modeling individual students' conceptual understanding

The form/function analysis for participants' understanding consisted of iterative analysis similar to Grounded Theory methodology (Charmaz, 2006; Glaser & Strauss, 1967). This analysis is differentiated from Grounded Theory most basically by the fact that the purpose of this specific analysis was not to develop a causal mechanism for changes in the students' conceptual understanding, but rather that it was used to develop a detailed model of students' conceptual understanding at given moments in time. For each interview transcript, I carried out an iteration of open coding targeted towards incidents in which the concepts of inverse and identity were mentioned or used. In this iteration, I focused on the representational vehicles used for the representational objects of identity and inverse and pulled excerpts that afforded insight into the correspondence that the participant was drawing between the representational vehicle and object in the moment. Along with the open codes, I developed rich descriptions of the participants' responses that served as running analytical memos. After the open coding, I carried out a second iteration of axial coding using the constant comparative method, in which open codes were compared with each other and generalized into broader descriptive categories. These categories emerged from the constant comparison of the open codes and were used to organize subsequent focused codes until saturation was reached. Throughout this process, I wrote analytical memos documenting the decisions that I made in forming the focused codes and, in turn, providing an audit trail for the decisions made in the development of the emerging categories. This supports the methodology's reliability (Charmaz, 2006).

Modeling proof activity

In the second stage of analysis, I first separated statements that conveyed a complete thought, initially focusing on complete sentences and clauses. I then reflected on the intention of each statement, focusing on prepositions and conjunctions that might serve to distinguish the intentions of utterances that comprise the sentence or clause. Following this, I compared these utterances to the model's constructs, focusing on which node an utterance might comprise. I constantly and iteratively compared each utterance relative to the overarching argument in order to parse out how the utterance served the argument in relation to other statements within the proof. For each proof, I then generated a working graphic organizer (i.e., a figure with the various nodes and how they are connected), including corresponding transcription highlighting the structure of the participant's argument. I then iteratively refined the graphical scheme to more closely reflect the structure of the argument as the participant communicated it. After this process, I completed a final iteration in which I compared the scheme to the participant's communication of the proof in its entirety to ensure that the model most accurately reflected the participant's communication of the proof. An expert in the field then compared and checked the developed Toulmin schemes against transcript of the interview in order to challenge my reasoning for the construction of the scheme, supporting the reliability of the constructions of the Toulmin schemes.

Relating conceptual understanding and proof

During the third and final stage of analysis, I focused on the participants' use of forms and functions within nodes of the Toulmin scheme, comparing the roles that specific forms and functions served in various nodes within the argument. I also focused on the shifts in which the participants' generated new, related arguments, specifically attending to concurrent shifts in forms and functions. I compared across arguments, looking for similarities and differences between the forms upon which the participant drew and the functions that the forms serve within the respective arguments. As in the previous stages, the analysis across conceptual understanding and proof centered on an iterative comparison of the patterns emerging across the analyses of the three participants' argumentation. In this comparison, I noted differences and similarities in the overall structures of Toulmin models for arguments. Further, I attended to the aspects of form/function relations that served consistent roles across similar types of extended Toulmin models. I continuously built and refined hypothesized emerging relationships through constant comparative analysis and memos. Through this process, I characterized constructs that unify the patterns found between the roles forms and functions of identity and inverse served across Toulmin schemes for the three participants.

Results

In this section, I discuss data from Tucker's second (midsemester) interview in order to demonstrate the construct of re-claiming that emerged during the third stage of analysis. I first discuss specific aspects of the form/function model of Tucker's understanding of inverse and identity. These codes of Tucker's conceptual understanding are relevant for discussing selected parts of his response to Question 7 of the protocol, which asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G (Figure 3). The reader might recognize the set H in question as the normalizer of the element h. This is can equally be thought of as the subgroup of elements that commute with h or the set of elements that fix h under conjugation, as the definition in the problem statement is structured.

Importantly, at the time of the interview, the notions of conjugation and normalizer had not been address in class. The students were familiar with proving whether subsets were subgroups, although, to this point in the semester, this often occurred with specific instantiations of subsets that the students could enumerate, rather than set notational definitions, although the participants were familiar with notation used to define subsets.

"Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 3. Asking participants to prove about the normalizer of *h*

For the sake of space, I have chosen to share three sub-arguments of Tucker's proof in response to question 7, specifically because they help demonstrate the construct of re-claiming without shifting between different participants' conceptual understanding and proof activity or shifting the focus of mathematical content too drastically. Further, I focus on Tucker's understanding of inverse so that the reader might gain a better sense of how Tucker drew on forms of inverse to serve specific functions in his goal-oriented activity. Tucker's response to Question 7 lasted about 40 minutes and involved several shifts between the three subgroup rules. Because of this, Tucker's proving activity in response to the prompt was modeled with Toulmin schemes for 11 sub-arguments, three of which are discussed in this article. These three sub-arguments provide a glimpse into Tucker's proving process in his effort to show that *H* satisfies the inverse and closure subgroup rules (that *H* contains each of its element's inverse and that the product under the group operation, *, of any two elements in *H* is also an element of *H*).

Form/function codes for Tucker's understanding of inverse

Tucker's discussion throughout the interviews supported the development of several codes for functions of inverse, three of which he used during the parts of his proof activity discussed herein: an "end-operating" function of inverse in which Tucker operates on the same end of both sides of an equation with a form of inverse, a "vanishing" function of inverse in which an element and its inverse are described as being operated together and are removed from an algebraic statement, an "inverse-inverse " function of inverse characterized by an element serving a function of inverse in relation to its inverse, and an "inverse of a product" *function* of inverse in which the inverse of a concatenation (or product) of elements is the reverse order of the inverses of those elements (e.g., $(g*h)^{-1} = h^{-1}*g^{-1}$) . Throughout his proof activity in these excerpts, Tucker draws on the "letter" form of inverse to serve these functions. Although these functions of inverse are consistent with ways inverses are typically used in formal mathematics, Tucker drew on these functions with varying fidelity to their formal treatment depending on the problem contexts in which he was engaged. Tucker continually demonstrated that he was able to use the "letter" form of inverse to serve these functions of inverse in order to appropriately manipulate equations and maintain logically consistent equations. When considering these functions of inverse relative to the Toulmin models of Tucker's proof activity, the functions tended to serve as warrants within Sequential layouts, connecting equations (serving as data) to logically equivalent equations (serving initially as a claim, then as data for the next claim in the sequence). This is evident in two of the three sub-arguments discussed in this article.

Proving that *H* satisfies the inverse subgroup rule

As mentioned, I will discuss three sub-arguments of Tucker's proof that help to explicate the construct of re-claiming. During his response to Question 7, Tucker read over his work saying, "I- you know what I might do actually?" (line 1078). He then explained

So, right now, we have *g* star *h* star *g* inverse is equal to *h*. We want to get to somewhere that looks like- … Want to show. *g* inverse star *h* star *g* is equal to *h*. In order for the inverse of *g* to satisfy this (points to definition of *H*) right here. Cause that's what you do when you put in the *g* inverse. (lines 1084-1086).

With this excerpt, Tucker began a subargument (Figure 4) of his broader, overarching proof for Question 7 in which he attempted to show that the set *H* contains the inverse of each of its elements. He began with the equation used to define *H*, saying, "right now, we have *g* star *h* star *g* inverse is equal to *h*" (line 1085), which serves as initial data (Data1.1) for the argument. He then described wanting to show that $g^{-1}*h*g = h$, which serves as the claim in the subargument (Claim1). He supported this claim by explaining that this goal means that g^{-1} satisfies the given equation, saying, "Cause that's what you do when you put in the *g* inverse" (line 1087). This warrants the claim by reflecting Tucker's previous activity in which he replaced *g* in the equation used to define *H* with its inverse and drew on the "inverse-inverse" function of inverse to rewrite the equation ($g^{-1}*h*g = h$). Although he had discussed the need to show that the set *H* satisfies the inverse subgroup rule earlier in his response, this the first time during his response that Tucker outlined a plan for demonstrating that the set contained inverses. This constitutes a shift in Tucker's description of what it would mean for the set *H* to contain inverse elements. Specifically, he anticipates manipulating the definition of *H* to result in the same equation he had obtained by substituting g^{-1} into the definition of *H*.

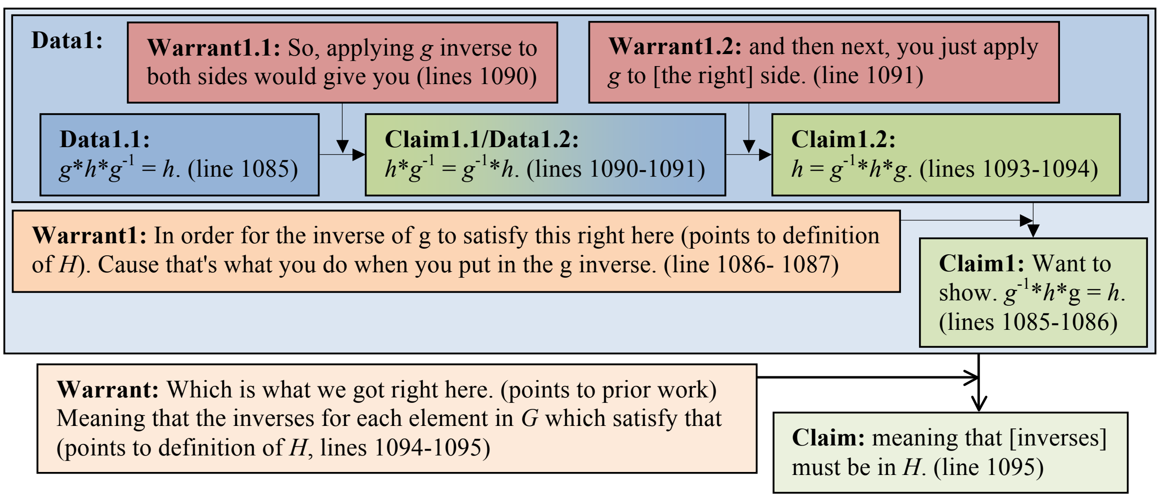

Figure 4. Tucker's inverse subproof in response to Interview 2, Q7

Tucker then continued, explaining how he might manipulate the first equation so that it looks like the second equation. Tucker began by left-operating with g^{-1}, saying, "let's apply the *g* inverse to that. So, applying g inverse to both sides would give you *h* star *g* inverse is equal to *g* inverse star *h*" (Warrant1.1, lines 1089-1091). This process comprises a warrant that draws on the "end-operating" and the "vanishing" functions of inverse to support the claim that a new equation (Claim1.1/Data1.2) can be produced. This equation then serves as data as Tucker describes right-operating with *g* to produce the equation $h = g^{-1}*h*g$ (Claim1.2). Similar to the left-operation with g^{-1}, this draws on the "end-operating" and "vanishing" functions of inverse to warrant the new claim. However, this action also subtly draws on the "inverse-inverse" function of inverse in that Tucker is using the element g as the inverse of its own inverse in order to cancel the g^{-1} on the right end of the left-hand side of the equation. Tucker then interpreted the result of this activity, saying, "Which is what we got right here. Meaning that the inverses for each element in *G* which satisfy that (points to definition of *H*), mean that must be in *H*" (lines 1093-1095), which comprises a warrant and claim for the overarching argument that *H* contains the inverses of its elements.

Tucker's work in this instance exemplifies the construct of re-claiming (Figure 5), which I define as the process of reframing an existing claim in a way that affords an individual the ability to draw on a specific form and the functions that it serves in meaningful (to the student) ways to support the new claim, which the student is then able to connect back to the original claim. In this study, it was often the case that re-claiming occurred when a participant was asked to prove or disprove a general statement and, in response, interpreted the general statement using a specific form to produce a new claim in terms of this form. For instance, in this example, Tucker substituted the "letter" form of inverse into the definition of *H* to produce the equation $g^{-1}*h*g = h$. An important part of successfully re-claiming is the consistency between the original claim and new claim. In this case, Tucker's new claim successfully reflects the assumption that g^{-1} satisfies the definition of the set *H*. The individual must also be able to interpret any possible hypotheses or assumptions of the original claim with respect to the new form upon which he or she draws. Again, Tucker successfully uses the assumption that *g* satisfies the definition of *H* and uses this as initial data in his construction of an argument. Once the individual generates appropriate initial data from the given hypotheses and assumptions, he or she is then able to draw on the new form to serve specific functions, which affords the development of meaningful argumentation toward the new claim. The various functions of inverse that Tucker was able to draw on to manipulate the equation in the initial data allowed Tucker to produce meaningful connections between the initial data he generated and his re-claimed claim. Finally, after supporting the new claim, the individual should be able to provide a warrant for how or why this claim supports the original claim.

Proving that *H* satisfies the closure subgroup rule

Among the other sub-arguments that Tucker generated during his response to Question 7, he produced two sub-arguments to show that *H* satisfies the closure subgroup rule which also help demonstrate re-claiming. Tucker describes this subgroup rule, saying, "[we] have to prove that all of those star themselves will yield you back another one" (line 762), which constitutes Claim of this sub-argument (Figure 5). He follows this by describing his approach, saying,

> *a* be an element of *H* and *b* be an element of *H*, then by definition of the set, *a* star *h* star *a* inverse is equal to *h* and *b* star *h* star *b* inverse is equal to *h*. And it looks like we can just, like, set these equal to each other. So, *a* star *h* star *a* inverse is equal to *b* star *h* star *b* inverse. See, what we're trying to do here is prove that *a* star *b* is also in *H* to prove that there's closure. That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out. Trying to show that *a* star *b* is an element *H*, which means *a* star *b* (mumbles, writing) - I'm just gonna need to write star between that - star *h* star *a* star *b* inverse is equal to *h*. So that's what we gotta, um, we need to prove. Um- Okay. I think it would be easiest to rearrange (points to $a*h*a^{-1} = b*h*b^{-1}$) this so that we get that (points to $(a*b)*h*(a*b)^{-1} = \mathrm{h}$). Hm. That's a good question. I don't know how we're supposed to do that, though. (lines 764-775)

This excerpt provides transcript of the entirety of this sub-argument, except for Tucker's initial claim. Notice that Tucker begins by fixing two elements of *H*, *a* and *b*. These serve as initial data (Data1) in a sequence in which Tucker draws on the definition of the set to warrant (Warrant1) the generation of two equations: $a*h*a^{-1} = h$ and $b*h*b^{-1} = h$

(Claim1/Data2). These equations then serve as data in order for Tucker to generate a third equation ($a*h*a^{-1} = b*h*b^{-1}$) comprised of the left-hand sides of the first two equations set equal to each other, serving as a second claim that is warranted by the phrase "it looks like we can just set these two equal to each other" (lines 766-767). Together, these data, warrants, and claims follow the structure of a sequential argument that serves as data that Tucker indicates should eventually support the claim that the element $a*b$ must satisfy the equation used to define H.

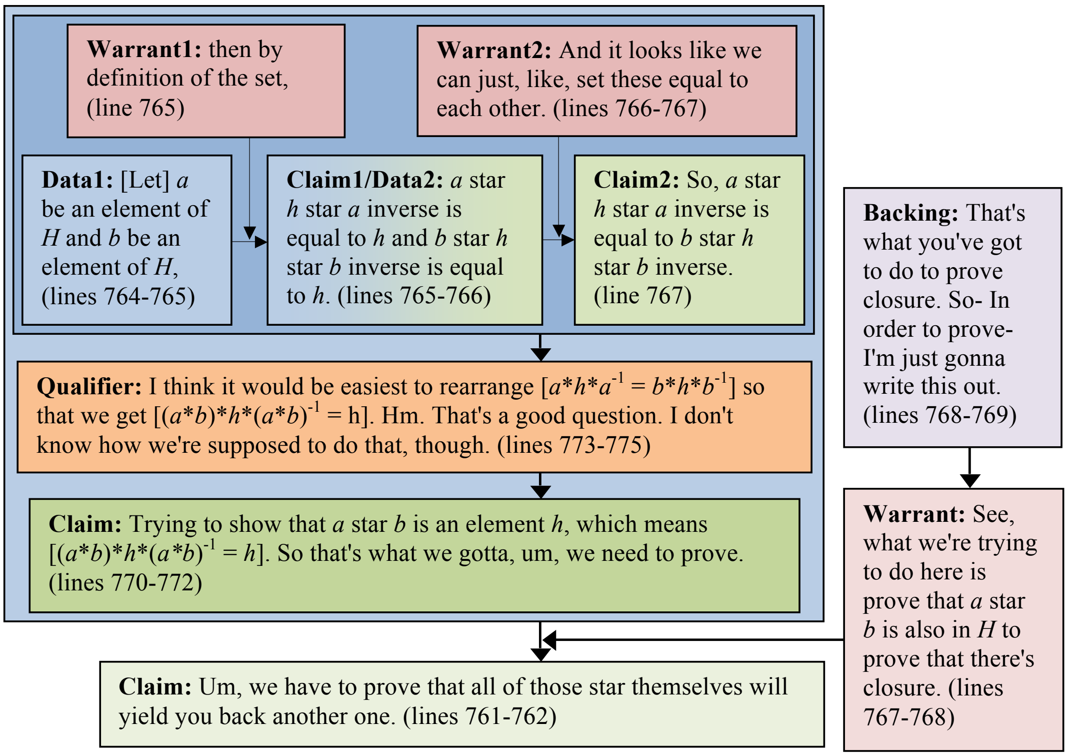

Figure 5. Tucker's first closure sub-argument in response to Interview 2, Q7

Tucker follows this initial sequential argument by rephrasing the original claim to reflect the definition of the set H. Specifically, similarly to Claim1/Data2, Tucker replaces g and g^{-1} with the algebraic statement "$(a*b)$," yielding the equation "$(a*b)*h*(a*b)^{-1} = h$." This shifts the goal of the proof from a more general description of closure to an algebraic framing that reflects the data that Tucker has produced. He then provides a series of statements that qualify, warrant, and back the embedded argument to support this new claim. First, he alludes to rearranging the equation "$a*h*a^{-1} = b*h*b^{-1}$" to "get that" while pointing to the written "$(a*b)*h*(a*b)^{-1} = h$," but admits that he is unsure how he might do this, which qualifies his argument. Tucker then provides a warrant for how his data might serve the claim that elements in H satisfy closure, saying, "See, what we're trying to do here is prove that a star b is also in H to prove that there's closure" (lines 767-768). He adds, "That's what you've got to do to prove closure. So- In order to prove- I'm just gonna write this out" (lines 768-769), which serves as backing for this warrant. Tucker's argument then stalls as he again expresses his uncertainty in the qualifier, saying, "This is a little trickier. I'm open to suggestions here" (line 784). At this point, Tucker has reframed the original claim that he has set out to prove and generated an equation that might serve as initial data in this argument, but reaches an impasse as he is unsure how he might be able to connect the two equations.

Later in his response to Question 7, Tucker returned to the closure subgroup rule, making a new sub-argument (Figure 6) to show that H satisfies the closure rule. Tucker initially indicated that he is still unsure of how to prove that the set satisfies closure. When asked what tools he could use, Tucker replies, “existence of inverses and identity. I don't know” (line 1148). The interviewer reminds Tucker that associativity holds and suggests, “Or you could just un-group stuff” (line 1154). This likely provided Tucker with some insight into a possible approach to the proof, as he said, “And, like, kinda like modify this right side as well, or- (4 seconds) Okay, so, if we bring that over-” and began writing. He then began this sub-argument by saying, “So, we know that a works and b works” (line 1163), which serves as the first data (Data1) in support of the claim, “We wanna show that $a*b$ works” (line 1164, Claim). It is likely that Tucker is using the word “works” here to indicate that they satisfy the definition of H. This sense is supported by Tucker generating the equations $a*h*a^{-1} = h$ and $b*h*b^{-1} = h$. Tucker then provides an argument that reflects his anticipated goal in the qualifier of the prior sub-argument, which he had described, saying, “I think it would be easiest to rearrange [$a*h*a^{-1} = b*h*b^{-1}$] so that we get [$(a*b)*h*(a*b)^{-1} = h$]” (lines 773-774).

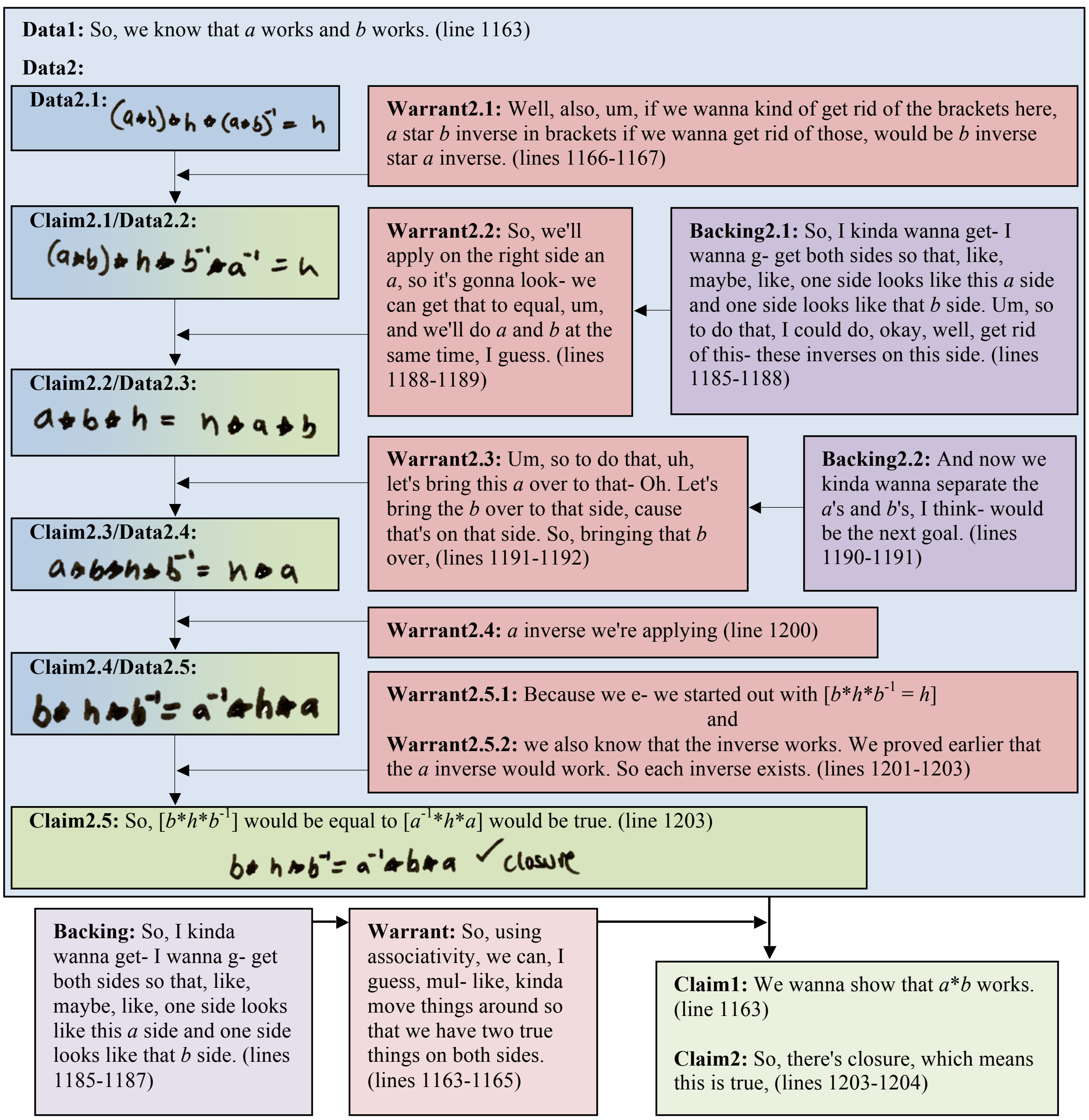

Figure 6. Tucker’s second closure sub-argument in response to Interview 2, Q7

Tucker describes how he expects to approach the proof, saying, “So, using associativity, we can, I guess, mul- like, kinda move things around so that we have two true things on both sides” (lines 1163-1165), which serves to warrant Tucker’s re-claiming. He then draws on the “inverse of a product” function of inverse (Warrant2.2) to support rewriting the equation $(a*b)*h*(a*b)^{-1} = h$ (Data2.1) as $(a*b)*h *b^{-1}*a^{-1} = h$ (Claim2.1/Data2.2). Tucker draws on this new equation to serve as data for developing yet another equation, first backing his activity by describing his reasoning for changing the equation, saying,

> So, I kinda wanna get- I wanna g- get both sides so that, like, maybe, like, one side looks like this *a* side and one side looks like that *b* side. Um, so to do that, I could do, okay, well, get rid of this- these inverses on this side. (lines 1185-1188)

This supports a strategy that it seems Tucker anticipates helping to eventually generate the desired equation. Tucker generates the new equation by drawing on the “end-operating” and “vanishing” *functions* of inverse to simultaneously remove the a^{-1} and b^{-1} from the right end of the left-hand side of the equation and concatenate *a* and *b* on the right end of the right-hand side of the equation, resulting in the new equation $a*b*h = h*a*b$ (Claim2.2/Data2.3). In doing so, he warrants his activity by saying, “So, we'll apply on the right side an *a*, so it's gonna look- we can get that to equal, um, and we'll do *a* and *b* at the same time, I guess” (lines 1188-1189).

Tucker continues by drawing on the same *functions* of inverse to remove the *b* that he had just concatenated on the right-hand side of the equation and concatenate b^{-1} on the left-hand side of the equation, essentially undoing part of his activity in which he generated the equation in (Claim2.2/Data2.3). He explains,

> And now we kinda wanna separate the *a*'s and *b*'s, I think- would be the next goal. Um, so to do that, uh, let's bring this *a* over to that- Oh. Let's bring the *b* over to that side, cause that's on that side. So, bringing that *b* over… (lines 1191-1192)

This serves as warrant and backing for Tucker to generate the equation $a*b*h*b^{-1} = h*a$ (Claim2.3/Data2.4). Tucker immediately follows Claim2.3 by saying, “*a* inverse we’re applying” (line 1200), which warrants the equation $b*h*b^{-1} = a^{-1}*h*a$. Throughout this entire data-warrant-claim sequence, new equations are generated by Tucker’s manipulation of the previous equation, drawing primarily on the “end-operating” and “vanishing” *functions* of inverse, the “inverse of a product” *function* of inverse, and also, implicitly, the “inverse-inverse” *function* of inverse. Having generated the equation $b*h*b^{-1} = a^{-1}*h*a$, Tucker interprets his work, saying, “…we started out with this and we also know that the inverse works. We proved earlier that the *a* inverse would work. So each inverse exists. So, that would be equal to that would be true. So, there's closure, which means this is true” (lines 1201-1204). In the first part of this excerpt, Tucker provides two warrants (Warrant2.5.1, Warrant2.5.2) that support the claim that the equation $b*h*b^{-1} = a^{-1}*h*a$ is true, seemingly drawing on the equation $b*h*b^{-1} = h$ in Data1 and his prior sub-argument showing that *H* satisfies the inverse rule for subgroups, which supports the equation $h = a^{-1}*h*a$. However, Tucker does not explicitly draw on these two equations or any sense of transitivity, instead saying “we started out with this” (lines 1201-1202) while pointing to the statement $b*h*b^{-1}$ and “*a* inverse would work. So each inverse exists” (lines 1202-1203), while pointing to the statement $a^{-1}*h*a$. Tucker finally concludes, “So, there’s closure, which means this is true” (lines 1203-1204, Claim2).

In this sub-argument, Tucker’s re-claiming activity is slightly different from the prior sub-argument. As before, Tucker begins by drawing on a specific form of inverse to reframe the original claim that H is closed under the operation (*). He also generated initial data from the hidden hypotheses of the original claim (considering two elements that are elements of H; Data1). However, the data that Tucker chooses to manipulate in order to reach some conclusion in the Embedded, Sequential Toulmin scheme is the assumption that $a*b$ satisfies the definition of H. He then develops a chain of reasoning that leads back to an equation that Tucker recognizes as verifiably valid based on his assumption in Claim1 that a and b each satisfy the definition of H. This reflects a not-uncommon proof approach in which the prover assumes the result that he or she intends to show and deductively shows that the result is equivalent to the assumption(s) he or she is able to make about the conjecture. Still, Tucker's activity reflects a type of re-claiming in which he was able to draw on a specific form to generate a new claim and initial data. From this claim and data, Tucker leveraged specific functions of inverse to demonstrate that the initial data and new claim were logically equivalent. Tucker's work is warranted by his initial claim that he would try and generate and equation that was true from his assumption that $a*b$ satisfied the definition of H.

Conclusions

This discussion of Tucker's proof activity should afford a sense of the various facets involved in re-claiming. Specifically, in re-claiming, it is not sufficient, to only reframe a claim. Rather, one must likely also reframe its related (often hidden) hypotheses. These aspects of reclaiming reflect the frequently taught proof mantras of "what do I know?" and "what do I want to show?" In this case, Tucker describes needing to show that $g^{-1}*h*g = h$ and begins with the equation $g*h*g^{-1} = h$, which reflects the assumption that g satisfies the definition of H. In the context of the form/function framework, these restated hypotheses serve as initial data (drawing on a specific form of identity or inverse) in a new argument in which the participant is able to draw on the form of identity or inverse with which the data is reframed to serve appropriate functions of identity and inverse in support of the new claim. The individual should then be able to reason that this new argument supports the original claim. In this sense, Re-Claiming provides a type of proof activity in which an individual's conceptual understanding (forms upon which an individual draws and the functions that these forms are able to serve) informs his or her proof approach. Specifically, the access to a form that is able to serve specific functions affords the individual an opportunity to generate a meaningful argument that he or she would likely not have been able to produce without Re-Claiming the initial statement. This activity is not necessarily an inherent necessity of a given conjecture, but rather depends on the individual's understanding in the moment.

The current research was constrained by several factors. First, my focus on three students' responses to individual interview protocols limits analysis of the relationships between conceptual understanding and proof activity, warranting further analysis of different participants' conceptual understanding and proof activity. Also, although this analysis was informed by the broader contexts of the classroom environment, the focus on the individual interview setting affords insight into a specific community of proof in which argumentation develops differently than in other communities. For instance, the structure of the interview setting necessitated that participants developed their arguments solely relying on their own understanding in the moment and for the audience of a single interviewer. My early observations of and reflections on the development of argumentation in the classroom and homework groups included the mutual development of argumentation in which participants' argumentation was informed by their interactions. Accordingly, analysis of the other collected data is warranted.

This research contributes to the field by drawing on the form/function framework to characterize students' conceptual understanding of inverse and identity in Abstract Algebra.

This affords insight into the forms upon which students participating in the TAAFU curriculum might draw as well as the various functions that these forms are able serve. The broader research also contributes to the field by providing several examples of how Aberdein's (2006a) extension of Toulmin's (1969) model of argumentation might be used to analyze proofs in an Abstract Algebra context. Further, this research draws attention to an aspect of the relationships between individuals' conceptual understanding and proof activity. These results situate well among the work of contemporary mathematics education researchers. For instance, Zazkis, Weber, and Mejia-Ramos (2014) have developed three constructs that also draw on Toulmin schemes to model students proofs in which the researchers focus on students development of formal arguments from informal arguments. These constructs provide interesting parallels with the three aspects of relationships between conceptual understanding and proof activity developed in the current research. Zazkis, Weber, and Mejia-Ramos (2014) describe the process of rewarranting, in which an individual relies on the warrant of an informal argument to generate a warrant in a more formal argument. However, the current research focuses more on the aspects of conceptual understanding that might inform such activity.

Moving forward from this research, I intend to analyze the data from other participants' individual interviews in order to develop more form and function codes for identity and inverse, affording deeper insight into the various form/function relations students in this class developed. Such analysis should also explore the proof activity of the other participants in the study, which would provide a larger sample of proof activity, in turn affording new and different insights into the relationships between mathematical proof and conceptual understanding. I also intend to analyze the sociomathematical norms and classroom math practices within the classroom. This will afford insight into the sociogenesis and ontogenesis of forms and functions at the classroom and small group levels in order to support and extend the individual analyses – which are focused on microgenesis – in the current research.

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