JOHN'S LEMMA: HOW ONE STUDENT'S PROOF ACTIVITY INFORMED HIS UNDERSTANDING OF INVERSE

Recent discussions in the field have explored proofs' explanatory power. Such research, however, focuses on how a written proof might convey explanation. I present a conjecture that individual proof activity (the development of proofs) might, itself, have explanatory power. I then discuss one student's (John's) activity related to proving that the centralizer for a fixed element in a group (the set of elements that commute with the given element) is a subgroup and how this activity informed his understanding of inverse. During an individual interview, John developed a lemma claiming that the left- and right- inverses of an element are the same element, his proof of which contradicted his previous ways of thinking about inverse. I analyzed John's proof activity using Aberdein's (2006) extension of Toulmin's (1979) model for argumentation in order to better organize his activity, providing an example of how proof activity might itself be explanatory.

Key words: Proof, Abstract Algebra, Inverse, Toulmin model

Educational researchers have clearly established the importance of exploring and discussing students' engagement in and understanding of mathematical proof (Hanna, 2000; Weber, 2010). Indeed, according to Harel and Sowder (2007), "No one questions the importance of proof in mathematics, and in school mathematics" (p. 806). This presentation contributes to this body of work by investigating successful proof activity adopted by one student. The data used in this article are drawn from a larger study of undergraduate students' proof activity and understanding in an abstract algebra context. This research focuses on modeling students' proof activity and conceptual understanding of inverse and identity, investigating relationships between the two. I chose the case presented here to give insight into how proof activity might inform understanding of the concepts used in the proof. This research outlines how one student's (John's) proof activity fostered meaningful change in his understanding of inverse in a group theory context.

Literature

Bell (1976) contends that mathematical proof should provide a sense of "illumination, in that a good proof is expected to convey an insight into why the proposition is true" (p. 24). Similarly, Almeida (2000) and de Villiers (1999) each claim that proofs should explain. These descriptions suggest that these researchers regard proofs as being inherently explanatory. This notion aligns with several other researchers (Hanna 1990; Mancosu, 2001; Steiner, 1978). Weber (2010) discussed a perspective regarding the notion of an explanatory proof that situates a proof's explanatory power relative to the proof reader. In his discussion, Weber describes "a proof that explains as a proof that enables the reader of the proof … to translate the formal argument that he or she is reading to a less formal argument in a separate semantic representation system" (2010, p. 34). This perspective is most clear in his critique of Steiner's discussion of mathematical proof, when he says, "Steiner treats an explanatory proof as a property inherent in the text of the proof rather than an interaction between the proof and its reader" (p. 34). Weber uses this point to draw distinctions between two representational systems that are used in the proof process: formal and informal. Formal representational systems are the signs, notation, and operations that one carries out in abstract thought, whereas informal representational systems largely rely on specific instantiations of concepts used as exemplars.

Considering this discussion, I see value in Weber's (2010) assertion that a proof conveys explanation only to the degree to which an individual is able to glean explanation from a written proof. The focus in the literature, however, is on individuals' understanding of the mathematical arguments in *others'* written proofs. This discussion neglects the individual's process of proving (developing proofs) which could provide an alternative perspective into proofs' explanatory power. The current investigation seeks to gain insight into the explanatory power of the proofs that one *constructs*, rather than the proofs that one might read.

Abstract Algebra curricula provide rich opportunities to explore students' proof activity due to their emphasis on student-generated proof. Early concepts, such as identity and inverse remain pervasive throughout the curriculum as more advanced concepts are defined using inverse and identity. Researchers have suggested that students' understanding of inverse in Abstract Algebra often build on their notions of multiplicative and additive inverse (Brown, et al., 1997; Hazzan, 1999) Novotna & Hoch, 1998) and in the context of symmetry groups (Almeida, 1999; Larsen, 2009, 2013). Larsen (2009, 2013) builds students' understanding in his Inquiry-Oriented Abstract Algebra curriculum (TAAFU) through students' experiences with symmetries of a triangle. Using both additive and multiplicative notation to relate the symmetries of the triangle, the students develop the group axioms from their exploration of triangular symmetries. Larsen (2009) describes the inverse and identity axioms as the most difficult "rules" for students to generate. By having students engage in the development of the group axioms via their work in S_3, Larsen provides a unique insight into students' generalizations from a specific case. Undoubtedly, students' experiences with inverses in other areas of mathematics inform their conceptions of inverse and identity in novel Abstract Algebra situations. It is interesting, then, that the inverse axiom is one of the two most difficult axioms for the students to develop from the Cayley table in Larsen's research. What aspects, then, of conceptual understanding of inverse would explain this difficulty? This question highlights a need to investigate students' developing notions of inverse and identity, specifically as they relate to the TAAFU curriculum. In this presentation, I focus on John's understanding of inverse.

In order to explore John's activity proving about inverse, I will use Toulmin's (1979) model of argumentation. Several researchers have adopted Toulmin models to document proof (Fukawa-Connelly, 2013; Pedemonte, 2007; Weber, Maher, Powell, Lee, 2008). This analytical tool organizes arguments based on the general structure of *claim*, *warrant*, and *backing*. In this structure, the claim is the general statement about which the individual argues. Data is a general rule or principle that supports the claim and a warrant justifies the use of the data to support the claim. More complicated arguments may use *backing*, which supports the warrant; *rebuttal*, which accounts for exceptions to the claim; and *qualifier*, which states the resulting force of the argument (Aberdein, 2006). This structure is typically organized into a diagram similar to a directed graph, with each part of the argument constituting a node and directed edges emanating from the node to the part of the argument that it supports (Figure 1).

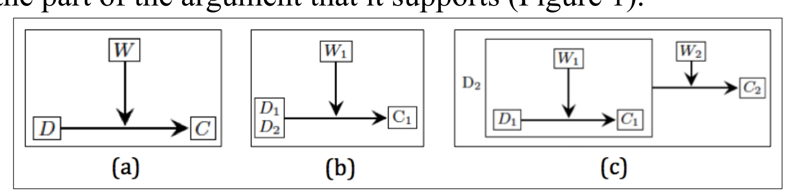

Figure 1: Visual representation of (a) Toulmin model, (b) Linked Toulmin model, and (c) Embedded Toulmin model (Aberdein, 2006, p. 211, 214)

Aberdein (2006) provides a discussion of how Toulmin models might be used to organize proofs, including several examples relating the logical structure of an argument to a Toulmin model. Aberdein includes a set of principles he to coordinate more complicated mathematical arguments in a process he calls *combining layouts*: “(1) treat data and claim as the nodes in a graph or network, (2) allow nodes to contain multiple propositions, (3) any node may function as the data or claim of a new layout, (4) the whole network may be treated as data in a new layout” (p. 213). Figure 1 shows two proposed combined layouts, Linked (Figure 1, b) and Embedded (Figure 1, c), that I are used in the current research. These combined layouts provide for multiple Data (Linked) and Data (Embedded) that are, themselves, claims in another argument.

Theoretical Perspective

As stated, this presentation makes use of data from a larger research project. The broader framing of this work draws on Cobb and Yackel’s (1996) interpretive framework for the Emergent Perspective, focusing on connections between sociomathematical norms (for proof) in the classroom community and individuals’ mathematical conceptions and activity, connections between classroom mathematical practices and individuals’ mathematical conceptions and activity, and relationships between individual mathematical conceptions and activity. This presentation focuses on relationships of the latter type, specifically, exploring proof activity that fosters change in conceptual understanding. Accordingly, the goal of this research is modeling John’s proving activity in an Abstract Algebra setting and identifying ways in which this activity informs his understanding of inverse in group theory. It should be noted, however, that John’s proving activity took place in an interview setting, which constitutes a community of practice distinct from, although informed by, the practices in the Abstract Algebra classroom of which John was a member. This view is consistent with Cobb and Yackel’s (1996) contention that,

> “…it is important to view the students' activity as being socially situated even in settings such as interviews, which are typically associated with psychological paradigms. The psychological analysis would then be conducted against the background of a social analysis that documents the interactively constituted situation in which the student is acting.” (p. 185)

The Emergent Perspective thus helps frame the current research as focused on an individual’s mathematical understanding and activity situated in an interview setting.

Data Collection and Methods

Data were collected in a Junior-level Inquiry-Oriented Modern Algebra course (Introductory Abstract Algebra). The course met twice a week, for 75 minutes per meeting, over fifteen weeks. The course instructor was an assistant professor in a mathematics department and taught using the Teaching Abstract Algebra for Understanding curriculum (TAAFU; Larsen, 2013). I conducted three (beginning, middle, and end of the semester, respectively), semi-structured individual interviews (forty-five to ninety minutes each) with seven participants. Each interview began by prompting the student to both generally and formally define identity and inverse. The interview protocol then sought to engage each participant in specific mathematical activity aimed to elicit engagement in proof or proof related activity. Throughout the interviews I kept field notes documenting participants’ responses to each interview task. I also audio and video recorded each of the interviews.

The data explored here come from John’s second (midsemester) interview. Specifically, I focused on John’s response to Question 7 of the protocol (Figure 2). This Question asked the participants to prove or disprove whether a defined subset H of a group G was subgroup of G. During the interview, it became clear that John was thinking about inverse in a specific way and that this way of thinking changed for John. I conducted a iteratively analyzed the video of John's response to question 7, attending to his argumentation related to the proof as well as his use of, notation of, and statements about inverse. In the first iteration of analysis I selectively transcribed sections of John's proof that developed his argument about whether H was a subgroup of G. These transcriptions were then used to generate a broad Toulmin model of his proof, outlining the major claim, data, and warrants used in his argument. A second iteration focused on each of the data in the larger argument, parsing out selected sub-arguments related to inverse that were used to validate each of the data in the larger argument. Two such sub-arguments emerged as focal points in the proof. The overarching Toulmin model was then modified to include these sub-arguments. Throughout the second iteration, specific aspects of how John thought about and used inverse emerged as important aspects of these sub-arguments. This prompted a subsequent iteration of analysis in which I characterized the various ways he talked about, used, and represented inverse throughout these parts of the proof. Finally, I coordinated one part of the Toulmin model with his ways of thinking about inverses, situating this coordination within the interaction between John and the interviewer.

> "Prove or disprove the following: for a group G under operation * and a fixed element $h \in G$, the set $H = \{g \in G : g*h*g^{-1} = h\}$ is a subgroup of G."

Figure 2: Interview 2, Question 7 asks participants to prove about the normalizer of *h*

Results

In this section, I first briefly detail aspects of John's conceptual understanding that inform an analysis of his proving activity. I then provide a general Toulmin model of John's proof to situate two specific parts of the broader proof: the first to highlight aspects of John's proof activity and understanding of inverse and also to provide an example of how Aberdein's (2006) extension of Toulmin models might be used to organize students' proof activity, the second to demonstrate how John's proof activity informed his understanding of inverse. I support each episode with excerpts of John's interview and a Toulmin model in order to convey a clearer understanding of John's thinking and proof activity throughout the proof. Finally, I will discuss the results and implications of this work for future research.

Throughout the interview, John worked with specific instantiations of groups as well as more abstracted representations of groups and their elements. For instance, John was very comfortable working with real numbers under addition and the group of symmetries of a triangle, evidenced by his frequent use of these groups when describing examples of concepts. John also referred to, proved about, or alluded to multiplication of real numbers, addition of real numbers, addition and multiplication of matrices, integers under addition, and the symmetries of a square. John defined identity and inverse using abstract, more general notation and completed two proofs using abstract notion. John described and used several specific ways of thinking about inverses. These included the cancellation law, self-inverses (elements of order 2 and the identity), the inverse of two elements when operated together ($(ab)^{-1} = a^{-1}b^{-1}$), that an inverse's inverse is the original element ($(a^{-1})^{-1} = a$), and that the identity is its own inverse. Early in the interview, when asked more generally about inverses, John alluded to situations in which a left-inverse and right-inverse might not be the same element. In written instances when an element was concatenated with or operated with its inverse, John either replaced the two elements with a representation of the identity or rewrote the entire statement and omitted the two elements.

John's approach to the proof in Question 7 began with a declaration that he would attempt to prove that H is a subgroup of G (Claim; C). He stated that it is typically easier to try to prove that something is a group and discover that it is not, rather than trying to prove that something is not a group. He had described his general approach for proving that a subset is a subgroup by determining that it satisfied each of the group axioms: H contains an identity (Data 1; D_1), H contains the inverse of each element in H (Data 2; D_2), H satisfies closure under the operation (Data 3; D_3), and H satisfies associativity (Data 4; D_4). From his discussion, the general Linked Toulmin model follows the format in Figure 3. John provided warrant for the broad proof by acknowledging that H satisfied the four group axioms (Warrant; W).

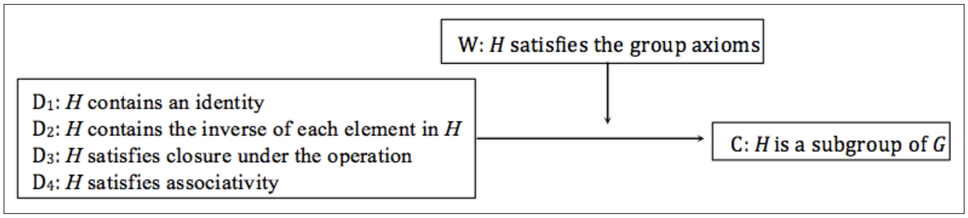

Figure 3: Linked Toulmin model for John's entire proof of Question 7

First, I focus on John's work related to D_1 in order to provide a better sense of how John proves with abstract representations of group elements. In order to verify that the set H contained an identity, John began his 45-second argument by "pretending" that the element g is the identity element. He then quickly said, "Then it really works. I know the identity exists." This is John's first statement of the claim that the identity element (of G) is an element of H ("the identity is in there (pointing at the letter H)". He then wrote the line "Let g, $h \in G$, um, and let g be the identity of G" followed by the equation "$g*h*g^{-1} = h$." On the next line, John rewrote the equation, substituting the letter e for g and g^{-1} and crossing through each e that he had written. As he wrote this, John stated, "and since the inverse of the identity is the identity and this (points to paper) is the identity, you get h equals h. Definition of identity. So we know the identity exists in the set." This sequence of statements is diagrammed in the Toulmin model in Figure 4.

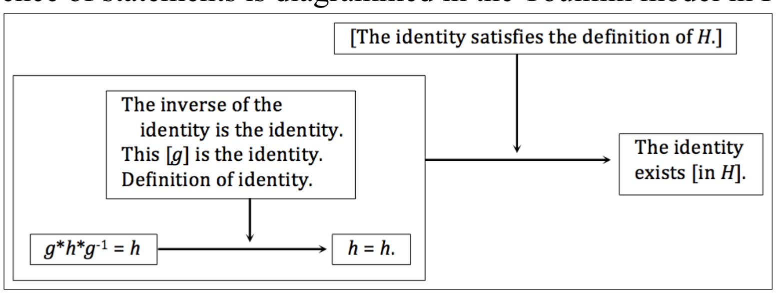

Figure 4: Embedded Toulmin model for D_1, showing an identity exists in H

Notice that John's proof begins with two assumptions, that g and h are elements of G and that g is the identity element of G. Following this, John assumes that g satisfies the relationship necessary for inclusion in H. He then conveys a chain of reasoning that results in the reflexive equation $h = h$. While each step in the sequence follows from the previous step and cites reason for the new statement, John fails to recognize that he has begun his line of reasoning with the statement he intends to prove, rather than beginning with the reflexive relation $h = h$ and deducing that the relation $g*h*g^{-1} = h$ holds. While this can be viewed as a problematic aspect of John's proof, consider the insight into John's ways of thinking about inverse that this excerpt provides. First, notice that John identified the inverse of the identity as the identity. Given the quickness with which he used this fact in the interview, one may infer that John is comfortable with thinking about inverse in this way. Notice also that John quickly rewrote the equation $e*h*e = h$ as $h = h$. This implies that John readily thinks of a composition of an element and the identity as merely just the element, while still citing the definition of the identity in his reasoning.

Next, consider John's development of a lemma used in developing D_2. While beginning his argument for D_2, John manipulated various elements in S_3 and seemed to come to a consensus that inverse elements were contained in *H*, saying, "Oh! I can do something with this." He then stopped and said, "Oh! That was assum- I can't do that, because that's assuming that left and right inverses are the same. I don't know." This prompted a discussion in which John described how he thought about g^{-1} as a left inverse and stated that he didn't know how to use g^{-1} if it's not on the left. The interviewer then asked John what would happen if left inverses were the same as right inverses. John replied, "Is that always true? That could be a thing that's always true." At this point in the interview, John left for a class, and returned later in the day to finish the proof. Upon his return, John announced that he had proved that left and right inverses were indeed the same element. Asked to explain his proof, John wrote out two equations, $a*b = e$ and $b*c = e$, concurrently stating that a is the left inverse of b and c is the right inverse of b. he then concatenated $a*b*c$ and grouped the concatenation as $a*(b*c)$ and $(a*b)*c$. From this, John wrote two lines: $a = a*(b*c)$ and $(a*b)*c = $ c followed by the line $(a*b)*c = a*(b*c)$, stating that this was true because groups are associative. Figure 5 provides a Toulmin model of this proof.

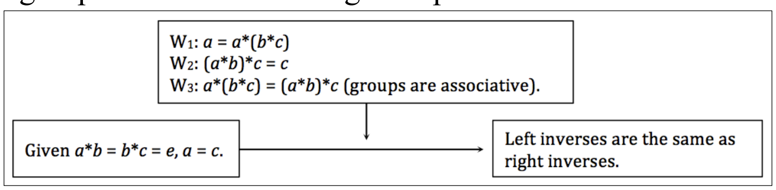

Figure 5: Toulmin model of John's Lemma

Discussion

John's lemma development was roused by his activity with examples in S_3 and his inability to progress in showing that the set *H* contained inverses of its elements. As he pointed out, he had assumed that left- and right- inverses were the same element, which he had previously stated to not always be the case. Prompted by the interviewer, John questioned whether this aspect of his understanding of inverse was valid and, following his proof of the lemma, declared that his new way of thinking about left- and right- inverses was valid. It is important to question whether John would have developed this lemma without prompting from the interviewer. This aspect of the situated interaction cannot be overlooked and must be accounted for. Similarly, the task setting itself prompted John to consider situations that he likely had not considered before (e.g., the definition of the centralizer of an element). Importantly, though, John attributed his changing notions of inverse to his validation of the lemma. This shows how the interview setting and John's willingness to question his own ways of thinking combined to afford an opportunity for him to inform his own understanding of the very concept he was proving about. From this, future research may be carried out exploring and classifying the different types of proof activities and interactions that afford similar opportunities. This work has contributed to the broader research literature by providing an extension of Aberdein's (2006) adaptation of the Toulmin model. To date, only one other research article (Wawro, 2011) has used Aberdein's Toulmin models to analyze students' arguments. The insight that these models lent in parsing out John's proofs warrants additional research in using Aberdein's adaptations.

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