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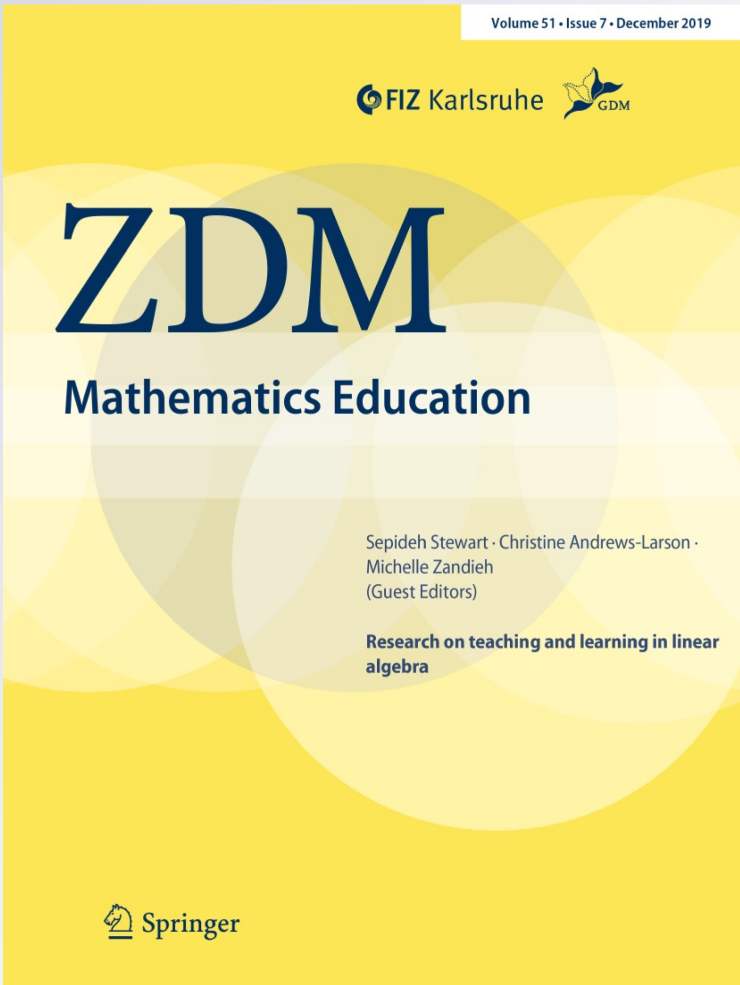

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ORIGINAL ARTICLE

Reflection on teaching linear algebra: examining one instructor's movements between the three worlds of mathematical thinking

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Abstract
Reflection is an important part of teaching and needs to be considered carefully. In this study, we examined a mathematics instructor's reflections on teaching linear algebra. The research team employed Tall's (How humans learn to think mathematically: exploring the three worlds of mathematics. Cambridge University Press, Cambridge, 2013) framework to track the instructor's movements between the three worlds of mathematical thinking. The instructor emphasized the importance of symbolic and embodied thinking and made sure the students were ready before entering the formal world. It became evident that moving the class between the worlds appropriately at certain moments was critical to students' understanding of the concept, however at times required much anticipation and careful preparation in advance as well as creating appropriate teaching resources. The instructor's dual role as a researcher and participant resulted a model of instructional decision making which was used to give further insights during his movements between Tall's worlds and afforded him unique opportunities for reflection and subsequent carefully calculated class interventions.

Keywords Tall's worlds · Linear independence · Model of decision-making · Reflection · IOLA

1 Background

Many researchers maintain that reflection is a critical aspect of teaching. For example, Dewey (1933) described reflection as "Active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (p. 9). Fund (2010) adds that "teachers need to develop particular skills, such as observation and reasoning, in order to reflect effectively and should have qualities such as open-mindedness and responsibility" (p. 680). According to Kolb (1984, cited in Mason, 2002, p. 124) "natural process of entering into situations as openly as possible, then standing back to reflect upon them, developing a framework or theory to account for them, and using this to inform and take action, resulting in yet further experiences". Mason (2002, p. 123) points out the importance of "recognition in the moment that something needs to be done (e.g., those moments when it is clear that most students are lost, that a diagram might be useful, or that you are about to present a difficult proof)".

Examining university mathematics instructors' reflection on teaching have had some attention over the past decade. For example, in a study by Johnson et al. (2013), the team focused on eliciting the mathematicians' reactions to an inquiry-oriented curriculum called Teaching Abstract Algebra For Understanding (TAAFU). By using a sequence of increasingly refined interviews, classroom data, and commentaries from the co-author participants, this study found three aspects of teaching important to the mathematicians, as a result of their post-implementation reflection, namely, the amount of coverage of course materials, the teacher's goals for student learning, and the intended role of the teacher.

As seen above, reflection has been researched steadily through a variety of frameworks, and has helped both teachers and researchers identify aspects of teaching deemed important by research mathematicians and teachers. Not only are mathematicians interested in teaching innovation and show rich reflection on their students, but these reflections can additionally benefit their teaching in various ways. For example, in two different studies, Hannah, Stewart, and Thomas (2011) and Pinto (2017) analyzed a mathematics instructor's reflections on his own teaching, using

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Schoenfeld's ROGs and TRUmath frameworks, respectively. Both of these studies suggested that reflection on one's own teaching can benefit the instructor, by creating "an increased awareness of his own orientations and goals" (Hannah et al. 2011, p. 983), and potentially acting as a useful tool for enhancing instructors' pedagogical awareness (Pinto 2017). Pinto additionally noted, "improvised teaching decisions should not be examined in isolation but rather as part of a flow of instructional moves that teachers make during a lesson" (p. 2220).

Similarly, Jaworski et al. (2017) investigated what a mathematics instructor did and thought while teaching mathematics using the Teaching Triad framework. The study found that despite teaching a primarily lecture-based course to a class of about 200 students, the instructor nonetheless showed sensitivity in attempting to relate to students' sociocultural needs as well as the problems he knew students typically faced in learning about convergence of infinite sums, and attempted to manage his classroom accordingly.

Another study attempted to elucidate certain teacher roles and goals for student learning. In particular, Burton (2004, p. 27) interviewed 70 mathematicians with a goal of building a model that approaches "the learning and teaching of mathematics as a meaning-making, rather than meaning-transferral, enterprise". Burton hoped to identify and bridge the gap between "how mathematicians themselves came to know and how they promoted learning in others" (p. 27).

Further research illustrated that educators think a great deal about how to bridge this gap. According to Winsløw et al. (2018, p. 63) "The tendency that University Mathematics Education (UME) teachers are increasingly engaging in efforts to this end—and that these efforts are now part of increasing number of studies of teaching *practice*, not only of teachers' *perspectives*—is reinforced by the bulk of recent CERME work in this area".

In addition to encouraging and investigating rich reflection on a variety of aspects of teaching mathematics, interviewing a teaching mathematician, and uncovering through these interviews and reflections those aspects of teaching and mathematics important to the instructor, this study additionally includes the mathematics instructor as part of the research team in order to deepen these findings. The first-named author's research, examining the mind of a working mathematician through reflections on teaching, indicates that involving mathematicians in every aspect of the research creates a rich collaboration and allows for a productive discussion of existing questions as well as creating new ones. Her research leverages mathematicians' reflections outside the classroom, through taking daily teaching journals followed by discussions in research meetings (e.g. Stewart and Schmidt 2017).

Although, research on pedagogy of linear algebra examining students' learning and conceptual resources has been steadily growing (e.g. Briton and Henderson 2009; Salgado and Trigueros 2015), the amount of research on instruction of linear algebra is equally important and is worthy of our attention. While some studies have explored instruction of linear algebra (e.g., Zandieh et al. 2017; Stewart et al. 2019a, b), more careful studies particularly those that focus on instructors reflecting on their own teaching of linear algebra and critically examining them are still needed.

In this study, the research team analyzed a mathematics education instructor's (the last co-author) reflections while teaching two sections of a first-course in linear algebra. To examine the instructor's reflections, the team situated this study within Tall's (2013) three-world model of embodied, symbolic and formal mathematical thinking.

2 Theoretical framework

Tall (2010) defines his three worlds of mathematical thinking as follows: The embodied world is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns…and other forms of figures and diagrams" (p. 22). Embodiment can also be perceived as giving body to an abstract idea. The symbolic world is the world of practicing sequences of actions which can be achieved effortlessly and accurately. In Tall's (2010, p. 22) words, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as number)". The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). In Tall's view (2013, p. 18), "formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used". He believes that the "formal mathematics can reveal new embodied and symbolic ways of interpreting mathematics" (p. 18).

Tall's theory describes the mathematical growth throughout an individual's life as a toddler to research mathematician. To examine the movements between Tall's worlds, in a study by Stewart et al. (2017), the team examined a mathematician's movements in teaching algebraic topology. The instructor noticed that students found the movement between embodied to formal the most challenging. Believing the struggle would stimulate mathematical growth in his students, this instructor "refused to give students proofs that were pre-packaged. More specifically, he desired to provide students with intuitions and pictures that would help them understand the conceptual nature of the proof and ultimately lead them to it" (p. 2262). In a different study Stewart et al. (2019a, b) examined a mathematician's (and co-author) movements between Tall's worlds while teaching eigenvalues and eigenvectors. This study encouraged teachers to explore "ways of motivating students to achieve a more holistic understanding of linear algebra concepts across the three worlds" (p. 7). In a study focused on students, Stewart (2018) suggested a set of linear algebra tasks in order to help students to move between Tall's worlds.

To further investigate the nature of these movements between the worlds, in the context of this study, we utilized the worlds to describe the instructor's treatment of the mathematical material in linear algebra. As this topic allows for many different representations in the embodied, symbolic, and formal worlds, it was ideal to employ this framework in order to analyze instructors' movements. We theorized that Tall's framework had the potential to track the instructor's movements between different modes of mathematical thinking.

In light of this theoretical framework, the research questions that guided this study were: (a) When did the instructor shift between Tall's (2013) three worlds of mathematical thinking, and (b) what was the rationale for these decisions to shift? (c) What were some of his challenges during these movements?

3 Methods

This qualitative narrative study (Creswell, 2013) took place at a research university in the US over an entire semester, during which a postdoctoral fellow and instructor (David) was teaching two sections of a first-year linear algebra course using the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et al. 2012). According to Creswell (2013), narrative researchers collect stories from individuals about their lived and told experiences, which shed light on their identities as individuals and how they see themselves. Narrative research is best for capturing the detailed stories or life experiences of a single individual or a small number of individuals. As Creswell states, in analyzing the qualitative data (e.g. interviews, observations), the researchers may take an active role and "restory" the stories into a framework that makes sense. Creswell (2013, p. 74) defines *restorying* as, "the process of reorganizing the stories into some general type of framework. This framework may consist of gathering stories, analyzing them for key elements of the story (e.g. time, place, plot, and scene), and then rewriting the stories to place them within a chronological sequence (Ollerenshaw and Creswell 2002)."

To help us generate this narrative framework, we first drew on grounded theory methodology to develop codes from David's teaching journals that he wrote. Merriam (2009) writes that the overall purpose of grounded theory "seeks not just to understand, but also to build a substantive theory about the phenomenon of interest. Narrative analysis uses the stories people tell, analyzing them in various ways, to understand the meaning of the experiences as revealed in the story" (p. 23). We sought to draw upon grounded theory methodology to analyze the topics and concepts that David chose to write about, generate appropriate codes, and investigate how they related to one another. Noting that Charmaz (2008) "argued that GT suffers from a split between subject and object, that is, it fails to account for the mutual construction between the respondent and the researcher; to account for the latter, she suggests that findings be presented as a story or a narrative" (Floersch et al. 2010, p. 4).

While we realize there is some inherent tension between including Tall's worlds codes a priori and the grounded theory approach, these two sets of codes corresponded to different scopes of analysis. We characterized Tall's worlds shifts as big picture shifts (macro), while the grounded theory codes (micro) provided greater detail that occurred between these shifts.

Schwandt (2007) notes that in grounded theory "experience with data generates insights, hypotheses, and generative questions that are pursued through further data generation" (p. 131). Our regular meetings with David encouraged him to reflect on his teaching experience and his journal entries through both the lenses of the research team's developing codes and his own views as an instructor. These reflections often led to David's implementation of a new plan of action in his linear algebra class, which in turn generated a new journal entry. We were able to continually refine our generated codes through this iterative process, enabled by David's dual role as researcher and teacher. This is also similar to Merriam's (2009) description of constant comparison, which she says "involves comparing one segment of data with another to determine similarities and differences. Data are grouped together on a similar dimension. The dimension is tentatively given a name; it then becomes a category. The overall object of this analysis is to identify patterns in the data. These patterns are arranged in relationships to each other in the building of a grounded theory" (p. 30–31).

Narrative studies also often collaborate with participants and actively involve them in the research (Clandinin and Connelly 2000). Nardi (2016) noted that "mathematicians have their own 'stories', their own ways of articulating how they make sense of their students' learning and their own pedagogical practices" (p. 366). In this study, we tell David's mathematics stories using his journals. These stories are laid out according to the narrative framework generated by the initial grounded theory methodology. In particular, David acted as co-author and helped the team analyze his own stories using the lens of Tall's worlds. Reflecting on the codes we collaboratively generated, David talked with the research team about the relationships he saw between these codes. We utilized these codes and relationships between them to help us define our narrative framework, which we could then use as a plot outline for our presented stories.

3.1 Participants and settings

David (a mathematics educator and the third author of the paper), acted as both a participant and as a member of the research team. As participant, David documented his instructional decisions throughout a semester of teaching linear algebra and provided continual member checking of the data throughout analysis and presentation. In his role as research team member, David worked alongside the team of two other mathematics education researchers. As co-author in this work, David continuously commented on the veracity of the writing of the other authors as well as making his own contributions. In addition, there is a reflexive relationship worth keeping in mind between David as research member and David as teacher. In particular, David reported on his teaching experiences in research group meetings, and these meetings allowed David to reflect on his own teaching in a concrete way by sharing his ideas with other team members. Conversely, perhaps as a result of these research meetings, David's teaching practices and decisions may have been influenced by these group discussions. For example, while David did not explicitly think about the three worlds while teaching, he was nonetheless aware of this theoretical perspective as a result of the weekly meetings, and this may have implicitly affected his teaching decisions.

3.2 Data collection

With some exceptions, David kept a journal of teaching reflections each week throughout the semester and met with the research team on a regular basis. David's reflections and weekly team meetings allowed for triangulation of data and gave multiple chances for him to share his reasoning about his teaching decisions. The team meetings were audio recorded and later transcribed as another source of data. For the purpose of this paper, we did not code or analyze these audio transcriptions, but we referred back to them to establish context for ourselves while reading his teaching journal.

3.2.1 The research meetings

We asked David to take regular journals without giving him any specific instructions on what to write. Our goal was to see a glimpse into his instructional processes and anything that he considered as noteworthy. We wanted to know as an instructor and mathematics educator what he reflected on. David made 12 journal entries. They ranged from one or two paragraphs to a page. The journals included his reflections on mathematics, teaching, and his perceptions of students. We have included some examples of these journal entries in the Results section. The team used these journals as a way of starting a conversation during the team meetings in which they asked more detailed questions regarding the mathematics in David's classroom and his pedagogical motives.

In total we had eight research meetings during the 15 weeks of Fall semester. Thus, we were able to discuss most of the linear algebra lessons that David covered during the semester. Sometimes we were able to meet weekly, so he only gave us information about the previous week (usually three lessons). Sometimes the gap was more (e.g. 2 weeks), so he would go over the events for that period of time. Each research meeting began with David giving a description of what happened in class over the past week or so. During that period the team would typically only listen to David, only asking clarifying questions. After about 15 min the team members would begin to freely ask questions. These meetings were carefully managed to focus the team's attention on David's reflections and comments in the moment. Specifically, discussions centered around the details of the mathematics involved in David's instruction. Throughout these meetings, David would attempt to recount his experiences of his in-class instruction, describe what decisions he made and why he made those decisions. During this part of each meeting, members of the research team regularly asked questions of David, probing him for clarity regarding his journal entries. Our conversations gave David another opportunity to reflect on each class and make new pedagogical decisions for the next class.

3.2.2 The tasks

During the semester in question, David's instructional approach was a mixture of a lecture and more inquiry-oriented methods. David was part of the team that created the IOLA material, making IOLA a natural choice for his teaching. This curriculum is based on Gravemeijer's (1999) curriculum design theory of Realistic Mathematics Education (RME), which informs curricula design to engage students in experientially real problem situations and rely on students' guided reflection on activity in the problem situation in support of the reinvention of more general mathematical relationships. The IOLA curriculum is comprised of three units and additional materials with each unit consisting of a sequence of four tasks.

In this article, we focused on David's reflections on his instruction implementing the fourth task in Unit 1, the Creating Examples task, as well as a lesson later in the semester that focused on using notions of basis to conceptualize linear transformations. The Creating Examples task is the culmination of the unit, which is intended to support students' development of vector arithmetic and linear combinations, span, linear in/dependence, and basis. Here, we will describe and unpack the mathematics of the Creating Examples task and two other instructional activities (the building sets task and the GSP task) that are important for our later discussion of David's journal entries. These three tasks are qualitatively different in important ways. Most importantly, the Creating Examples task is an instructional task that developed from the organized, iterative design of an entire research team over several years. Conversely, the building sets and GSP tasks are activities that David developed on his own in response to specific situations in his own instruction. These tasks are heavily structured and are centered around the instructor's (in this case, David) mathematics with frequent rhetorical and Socratic questions used to engage students' thinking and input into the trajectory of the discussion.

Note that all examples focused on various mathematical aspects of the topic of linear independence and dependence. We chose to focus on these tasks, since as David pointed out in his journals, he made shifts in reaction to perceived student feedback.

3.2.3 Task 1: linear independence and dependence: creating examples

The Creating Examples task asks students to complete five rows and two columns of a table by generating sets of vectors that satisfy given constraints (Table 1). Each row denotes a different number of vectors in a different real vector space. For each row, students are asked to generate a set of vectors that is linearly dependent (column 1) and another set of vectors that is linearly independent (column 2). Students are also asked to keep track of rules they develop to help produce the sets they generate. This activity is intended to support a connection between notions of span and linear in/dependence, leading to the development of the definition of basis.

3.2.4 Task 2: the building sets task

The building sets tasks paralleled the Creating Examples task in that David wanted the students to focus on the connections between and consequences of including a new vector in a set (vis a vis linear independence and span). Just after defining span in the class, David asked students to imagine beginning with an empty set and including a new vector in the set, one at a time. After imaginatively including a vector, he asked students to consider the possibilities of whether the span had increased. So, for instance, beginning with an empty set, David would ask students to imagine putting a vector in the set. He would then ask what the span of that set is. Often referring to their work toward defining span earlier in Unit 1 of IOLA, students might say that the span is a line. Typically, when a whole class of students is asked this question, one student will recognize that, if the vector in the set is the zero vector, then the span is just the origin (a point). After this, David then asked students to imagine a second vector being included in the set. This results in two possibilities: either the new vector increases the span of the set or it does not. In the case of two vectors, the second vector would need to be a scalar multiple of the first vector in order for the span of the set to remain the same. This process is then repeated with the inclusion of one new vector at a time and the same question of what the possible consequences might be about the span of the set (either it increases by one dimension or it stays the same). As we discuss in the results section, after implementing the Creating Examples task, David repeated this task, instead asking whether the linear (in)dependence of the set had changed.

Table 1 The main table from the creating examples task

	Linearly dependent set	Linearly independent set
A set of 2 vectors in $\mathbb{R}^2$		
A set of 3 vectors in $\mathbb{R}^2$		
A set of 2 vectors in $\mathbb{R}^3$		
A set of 3 vectors in $\mathbb{R}^3$		
A set of 4 vectors in $\mathbb{R}^3$		

3.2.5 Task 3: GSP

During the semester, David developed an activity using dynamic geometry software. This activity was intended to highlight linear transformations during a whole-class discussion. He designed a *Geometer's Sketchpad* (*GSP*) file to support this activity in advance of his in-class instruction, anticipating an unfolding discussion about linear transformations. David used *GSP* to produce two bases that shared an origin (Fig. 1d)—one using the standard $\mathbb{R}^2$ basis vectors (v_1 and v_2) and another using nonstandard basis vectors [$T(v_1)$ and $T(v_2)$]. Within the Cartesian coordinate system, David constructed a vector (v) and corresponding parallelogram with each side parallel to the two standard basis vectors (Fig. 1c). Following this, David constructed another parallelogram that is the same linear combination of the basis vectors in the second coordinate system (Fig. 1d). During the class discussion, David guided students through a discussion intended to determine the image of v under the transformation that mapped v_1 to $T(v_1)$ and v_2 to $T(v_2)$. David's goal was to leverage the preservation of linear combinations to support students' thinking about linear transformations.

Given our view of Tall's worlds, we characterized the Creating Examples task as symbolic. We identified the building sets task as embodied, and the idea of linear dependence

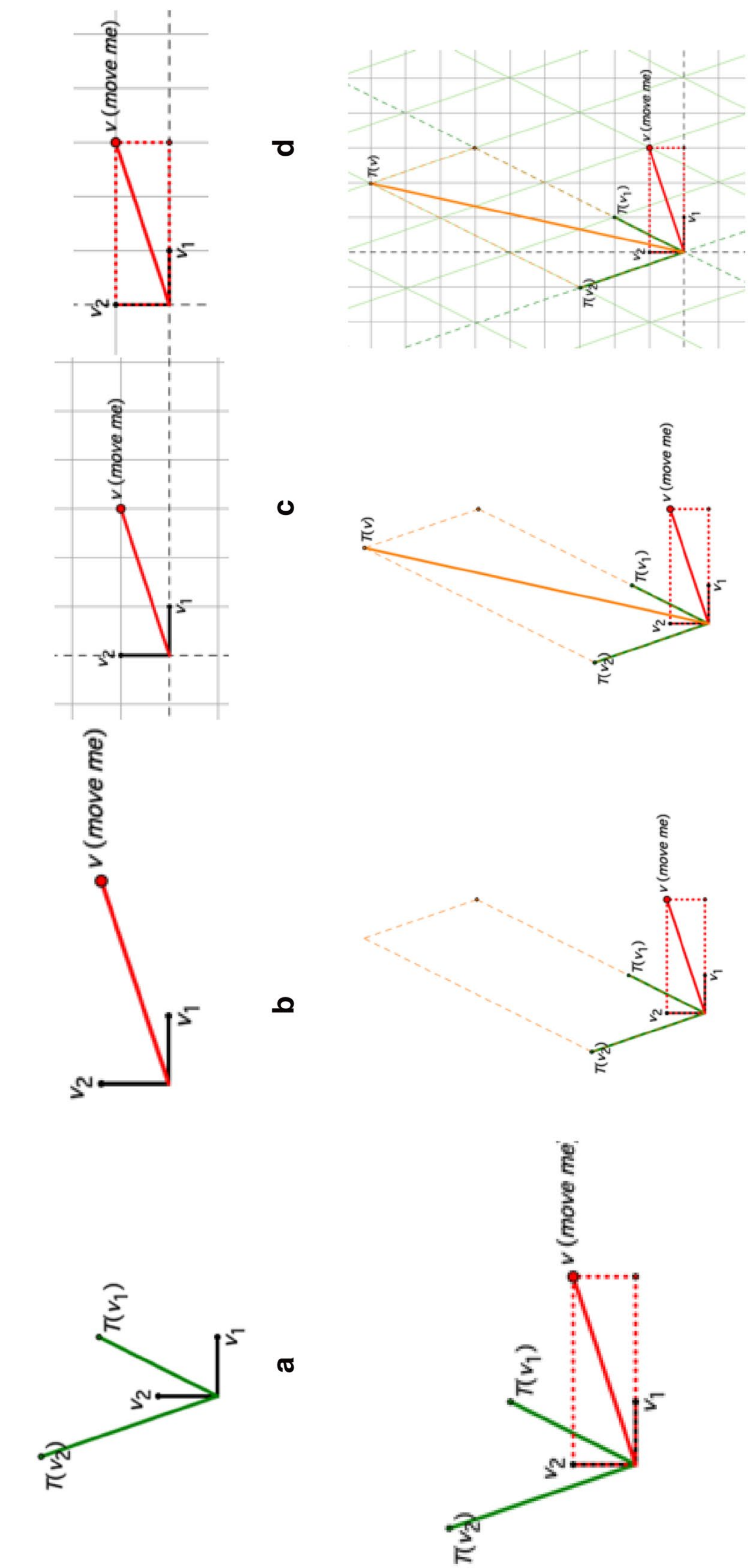

Fig. 1 Images from the Geometer's Sketchpad Lab

as redundancy as formal. Thus, in David's implementation of the building sets task, he utilized the embodied world as a teaching intervention to support this formal view of linear dependence. We also characterized the GSP demonstration as embodied.

3.3 Data analysis

In order to explore instructional decision making, the team examined David's teaching journals. The team then conducted a retrospective analysis of the journals following the methodology of narrative study (Creswell 2013). Specifically, the team iteratively coded the data, beginning with open coding (Strauss and Corbin 1998) that each member of the team conducted and brought together to compare. Given that we used David's journal as a central data source, all codes generated were aspects of David's reflection on his teaching experience. For example, codes generated under the Students tab in Table 2 were developed to explain David's personal perception of his students while teaching, and not the students' actual objective responses. These codes provided greater insight surrounding the circumstances of the instructor's moves through the worlds.

Through constant comparison of open codes, the team developed a set of focused codes that were iteratively refined through collective discussion. These codes are not mutually exclusive. The team then used these focused codes to categorize each sentence from the journals, disputing conflicts through an open discussion until each member of the team was satisfied. This process further refined the focused codes. These discussions resulted in a spreadsheet with each sentence from the journals coded for as many categories (themes) as the group deemed necessary for that section of transcript. Some of these codes are listed in Table 2.

The research team grouped similar themes with each other based on which aspects of the pedagogical process David was discussing. The broad categories included: Teaching, which describes codes in which he is describing what occurred in class; Students, which indicated that an excerpt referred to aspects of the students' actions, statements, questions, and behavior; Class Activities/Technology, which indicates that David was writing about the use of the various resources and technologies he had access to; Assessment, which indicated that an excerpt included an assessment of student thinking or a reference to assessment that occurred during instruction; Math, which differentiates instances in which David was explicitly writing about either the students' mathematics or his own; Reflection, which focused on the successes and failures of implementation toward the desired learning goals; and Tall's worlds, which focused on which of the three worlds David was drawing on in the moment. The codes that we focused on in this article are when David

Table 2 Focused codes from the iterative coding of the instructor's reflections

CODES	
Teaching (T): Describing what the instructor did in class	
Tasks	Tt
Developing ideas	Td
Response (formative assessment)	Tr
Real life problems	Trl
Pedagogical decisions	Tp
Students (S)	
Reactions to HW assignments	Sr
Class 1	S1
Class 2	S2
Student affect	Sa
Students asking questions	Ss
Class activities/technology (CA/T)	
White boards	CAwb
Dropbox	Cadb
Geometer's sketchpad	Cags
Group work	CAgw
Assessment (A)	
Tests	At
In-class/formative	Ai
Math (M)	
Instructor	Mi
Students	Ms
Reflection (R): Instructor reflecting on what he might have done differently or on the success/failure of implementation	
Students	Rs
Implementation	Ri
[Instructor's] Affect about classroom experience	Race
[Instructor's] Affect about study participation	Rasp
[Instructor's reflection] on self	Ro
[Instructor's reflection on] pacing	Rp
Comparing to prior Experiences	Rc
Tail's three worlds (W)	
Embodied	TWe
Formal	TWf
Symbolic	TWs
Shifting from embodied to formal	TWef
Shifting from formal to embodied	TWfe
Shifting from embodied to symbolic	TWes
Shifting from symbolic to embodied	TWse
Shifting from formal to symbolic	TWfs
Shifting from symbolic to formal	TWsf

discussed: teaching, focused on the tasks implemented in class (IOLA); teaching, focused on developing specific ideas in the class; teaching, when pedagogical decisions are made; statements about David's mathematics; statements about the students' mathematics; and reflections specifically addressing the students' successes and struggles in developing the intended mathematics.

In categorizing for Tall's worlds, we considered statements from the instructor's journals that referenced algebraic formulas or other symbolic manipulations as representative of the type of manipulations that take place in the symbolic world. We categorized statements from the instructor's journals that contextualized mathematics in terms of geometric tasks or leveraged dynamic imagery as representative of those processes in the embodied world. Finally, we classified statements from the instructor's journals that referenced linear algebra theorems, proofs, similar logical structures, or theoretical ideas from linear algebra as representative of the formal world.

We chose to present two specific instances where the instructor made adjustments to the instructional sequence in response to his students' difficulties.

4 Results

The goal of the study was to investigate David's shifts between Tall's (2013) three world model and his rationale for his pedagogical decisions during these shifts. The mathematical focus of the study was linear independence and dependence in linear algebra.

David utilized the codes developed by the research team to create an overview capturing the details of his teaching processes as evidenced by the research team's coding of the journal. While going through this process, he presented a model of decision-making that shifted from planning, to implementation, to reflection, and over again (see Fig. 2). This model was intended to reveal the essence of his teaching processes and captured the sequences of events that occurred throughout many of his teaching experiences.

We as researchers used the model to investigate the specific rationale for each of these shifts. We were interested in finding out as much information during each shift. The model gave us more detail about each of these moments and helped us describe each shift in a manner that was reflective of David's teaching as he saw it. Thus, despite the fact that David did not think about Tall's three world model explicitly while teaching, he was nonetheless able to describe his teaching in terms of the vocabulary of Tall's model without sacrificing the substance and authenticity of the reflections on his own teaching.

4.1 The model of instructional decision-making

In the proposed model (see Fig. 2), David presented the specific ideas he developed in the classroom (Td) and the pedagogical decisions he made (Tp) informed the tasks (Tt) in which he engaged the students. David then reflected on the students' activity (Rs), and explicated the insight this allowed him to gain about their mathematics (Ms). Following this, he then drew on his own mathematical understanding (Mi) to make sense of the students' mathematics in the context of his reflection on his own instruction (Ri). This act, in turn, informed his pedagogical decisions (Tp) and focus on which ideas to develop (Td) as well as a means of developing them through specific tasks (Tt).

We do not view this model as all-encompassing or even universal. Rather, we found this model to be consistent with David's narrative framing of his instruction while also incorporating codes from our team's analysis. Further, this cycle of decision-making can occur over a matter of days, such as during David's journal writing and debriefings with the research team, or in the moment, such as when an instructor reacts to student responses in whole-class discussions.

Note that Tall's worlds was the lens with which the research team viewed David's reflections on his teaching, whereas the model is a manifestation of David's own views of his reflections while teaching, albeit in terms of teaching codes the research team developed. The model was considered as a microscopic view of what took place during those moves and was used as a way of gaining more insight (see Fig. 3).

4.2 Narratives

In this section we used narratives to describe the analysis of two stories of David's teaching in the context of both the

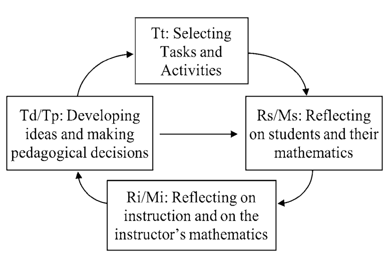

Fig. 2 The model of instructional decision-making

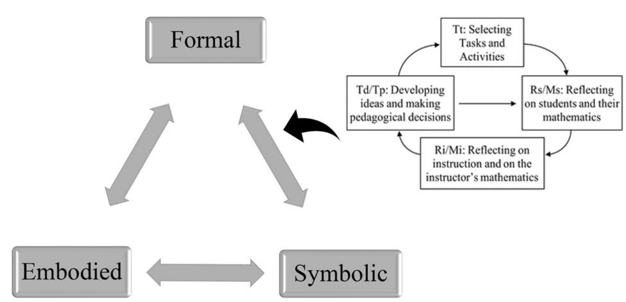

Fig. 3 The connection between Tall's worlds and the model

model David presented and the lens of Tall's worlds. We identified these stories due to the fact that we noticed a clear shift between at least two of Tall's worlds in the coding and in the journals. In this study we investigated the following two stories. Narrative (a) occurred September 2nd to September 9th, on three consecutive class days (see Tables 3, 4). Narrative b occurred on October 12 (see Table 5).

(a) Starting with the formal definition of linear independence and dependence, then shifting to give some symbolic and embodied motivation

David's intentions were to teach that linear combinations of linearly independent vectors are unique, while linearly dependent vectors are not. He gave a definition of linear independence, but apparently rushed through it. However, he faced difficulty reintroducing this topic 5 days later, therefore, made the pedagogical decision to implement two symbolic tasks—a quick algebraic manipulation and another that required deeper thought. The second involved two linear combinations of linearly independent vectors both equal to the same vector. "I set up two LCs of a LI set of vectors that equaled the same vector equations, rather than as matrix equations. Some students quickly noticed that if b = 0, then a and c must be 0." Reflecting on the students' mathematics in the moment, David wished he had recorded students' small group conversations.

David continued the investigation of linear independence with the Creating Examples task during the next class. The Creating Examples task is largely focused on students' generation of sets of vectors represented symbolically in order to support broader generalizations about linear independence and dependence. Despite the fact that Wednesday had felt like a successful (algebraic-based) discussion of linear independence, after he implemented the next task on Friday David realized the students had not connected their discussion on Wednesday to a sense of dependence as redundancy as he had hoped. In particular, David realized that several groups of students had not recognized impossible cases such as the request to create a set of three linearly independent vectors in a two-dimensional vector space. Reflecting on the mathematics involved, David felt that the students' difficulties with the symbolic Creating Examples task were, in his words, "a result of rushing through the definition of LI/LD and my failure to support deep geometric thinking about linear dependence."

Embodied world first to reach a new formal goal For David, this reflection brought to mind the embodied "building set" task he had previously created to teach the concept of "span". As a result, he decided to address this perceived problem by repurposing this task to focus instead on linear independence and dependence. That is, he decided to shift from the symbolic to the embodied world in order to support

Table 3 Journal entries on three consecutive class days

Friday, September 2nd	Wednesday, September 7th	Friday, September 9th
…Unfortunately, though, I was REALLY rushed at the end of class and only had about 5 min to introduce linear (in) dependence	This was a tough day because it had been 5 days since the last class. I planned on getting the students to recall their solutions to the getting back home problem, especially how this led to the definition of linear (in)dependence and I had 2 goals for the day: (1) I wanted them to see that if a set of vectors was linearly dependent, then there were an infinite number of solutions to the homogeneous equation and (2) I wanted them to think through how to show that linear combinations of linearly independent vectors are unique. The first goal was a quick algebraic manipulation. For the second, I wanted them to think through it, so I set up two LCs of a LI set of vectors that equaled the same vector (Mo = *b* = Mc, but written as vector equations, rather than as matrix equations). Some students quickly noticed that if *b* = 0, then *a* and *c* must be *0*. Others had really cool ways of thinking about *why a* and c had to be equal. I wish I had video of some of those small group conversations	I wanted students to complete U1T4 from IOLA. Each class finished the task, though some groups had some pretty serious reasoning deficiencies. For instance, very few groups in the first class realized the impossible cases. I think this is a result of lushing through the definition of LI/LD and my failure to support deep geometric thinking about linear dependence. I think I can help fix this on Monday by having the students do the "building set" task while focusing on LI/LD. We've done this task when talking about span and so I think they'll be comfortable with it, I just need to give them time to feel comfortable with thinking about linear dependence spatially

Table 4 Coding David's journal from Friday, September 9th

Excerpt	Tt	Td	Tp	Mi	Ms	Rs	Ri	TWe	TWs
I wanted students to complete U1T4 from IOLA	1		1						
Each class finished the task, though some groups had some pretty serious reasoning deficiencies					1	1			
For instance, very few groups in the first class realized the impossible cases					1	1			
I think this is a result of rushing through the definition of LI/LD and my failure to support deep geometric thinking about linear dependence					1		1	1	
I think I can help fix this on Monday by having the students do the "building set" task while focusing on LI/LD	1	1	1	1					
We've done this task when talking about span and so I think they'll be comfortable with it, I just need to give them time to feel comfortable with thinking about linear dependence spatially				1		1		1	

Table 5 Journal entry for Wednesday, October 12

Wednesday, October 12
I tried die approach that we talked about—to focus the importance of basis with considering the preservation of linear combinations and representing that preservation algebraically. It worked okay with my second class, but failed miserably with my first class. That was really frustrating. Some of the students got really hung up on whether the matrix itself was a transformation. I tried to quell it by pointing out the difference between an array of characters and a mapping from one vector space to another. The point that I was worried about was that the same transformation can be described using a different matrix if you name the input vectors relative to a different basis. Of course, I didn't say this explicitly to the students. Instead I tried to focus on the difference between a symbol and an individual's activity that relies on that symbol. The example I used was flunking of "$x+3$" as a mapping from $\mathbb{R}$ to $\mathbb{R}$. I focused on the point that there's nothing inherently functional about the algebraic statement "$x+3$," but tliat we can think of it as a function *if we* imagine *actually putting values into* the algebraic statement. The analog for me—that I intended for the students to pick up on—was that a matrix isn't a linear transformation until you actually do something with it, like multiply it times a vector. This seemed to be lost on some students in the first class, which was really frustrating. The thing that I think was so frustrating was that the issue seemed to blindside me out of nowhere. We have done an entire unit of them finding matrices that stand for LTs, but this hangs them up. It's one of those times when I took something as shared, but it wasn't shared—this is a thing ☺). So, the message of preservation of linear combinations got kind of lost and I went back to the drawing board to figure out how to talk about it. I had an old GSP file that I like to use to talk about preservation of LCs, so I dug it up to use

students' thinking about the formal idea of linear dependence as redundancy. Thus, David synthesized the symbolic and embodied worlds in order to help students approach the formal world. This reflection is essentially a conjecture that the students' difficulty with a symbolic task stemmed from a lack of experience with the embodied world. It additionally suggests that David valued embodied world thinking for the purpose of deepening his students' understanding of the formal concept of linear dependence as redundancy, and that David was re-evaluating his feeling that the desired aspects of linear independence and dependence had been adequately grasped by the students on the previous day.

As a result of David's reflections, he made the pedagogical decision to move students away from the symbolic emphasis of the Creating Examples task and toward the more familiar and geometrically rooted "building sets" task in the hopes that this would encourage students to leverage an embodied sense of the relationships between vectors within a linearly independent and linearly dependent sets. He noted that the students had previously completed this task with a focus on span, but this time, they would focus on linear independence and dependence. David further noted that he believed the students' familiarity with the task should help, and they just needed time to think about the new context of linear independence while going through a similar process of imagining the inclusion of more vectors in a set. Perhaps as a result of this work with linear independence and dependence, he decided to set a new goal of defining basis.

(b) Shifting between all three worlds while emphasizing the importance of basis with respect to linear transformations

Initially, David attempted to symbolically teach "the importance of basis with considering the preservation of linear combinations and representing that preservation algebraically." Here, David "aimed to draw out the need" for students to realize that a linear transformation is defined by images of basis vectors, and that column vectors of a matrix must form a basis to be invertible, which we view as a goal situated in the formal world. While considering this teaching goal during the research meeting immediately preceding this week, David discussed a way to algebraically demonstrate to his students the preservation of linear combinations under a linear transformation [e.g., $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$], specifically with a focus on the importance of the notion of basis for writing every preimage and image vector uniquely.

Formal to Symbolic Connection not made It seemed David had mixed success in introducing this topic. Reflecting on the prior meeting with the research team, David wrote that the approach he had discussed with us did not resonate with one of his classes. In particular, he stated that his approach worked satisfactorily for his second class, but "failed miserably" in his first class. This frustrated him, as students focused instead on whether the matrix itself was a transformation.

Symbolic to Embodied connection not made David reflected on his own mathematics to help him decide how to tackle this issue. He wrote, "the point I was worried about was that the same transformation can be described using a different matrix" under a different basis, but he "did not say this explicitly to the students." Perhaps due to this reflection, David made the pedagogical decision to address this issue by first pointing out the difference between a simple array of characters using a symbolic representation of a matrix, and a mapping from one vector space to another using an individuals'. To drive his point home, he additionally presented an analogy with the written statement "$x + 3$". In particular this expression is not inherently functional (symbolic), but it can be thought of as a function if one imagines placing values (embodied) into the algebraic statement. This analogy relies on a different symbolic system, but focuses on how an individual interacts with the symbols. Thus, David was attempting to tune his students into his own viewpoint that a "matrix isn't a transformation until you do something with it, like multiply it by a vector." This analogy between an isolated symbol and the activity sparked by that symbol is notable in that it is similar to the difference between symbolic and embodied reasoning. Unfortunately, the analogy did not go over as planned. In particular, David was quite frustrated that the analogy was lost on students in the first class, especially since they had "done an entire unit of them finding matrices that stand for LTs, but this hangs them up." David concluded that it was "one of those times when I took something as shared, but it wasn't shared." Thus, as his message of preservation of linear combinations had been lost, as part of his reflections he stated he wanted to go back to the drawing board and "hit a reset button" to clear the air.

Embodied to symbolic connection-the reset button Reflecting again on what David experienced as student difficulties about the formal concept of basis in the context of linear transformations, David made the pedagogical decision to develop and implement an interactive embodied *Geometer's Sketchpad* demonstration/class discussion. As part of this reflection, David realized he had some idea of how the students would respond to various aspects of this task, so he developed incremental stages of geometry using *GSP* accordingly. As a result of this planning ahead, in class, he was able to respond to the students' reactions using these preconstructed contingencies. In his implementation of the *GSP* task, he at first just displayed a vector as a line segment, and asked his students to name it. In response, his students noted they needed to see the axes to name the vector appropriately. Reacting to this stated need, he therefore displayed a standard basis and an associated lattice, so that the students could estimate linear combinations of the basis vectors. Afterwards, he placed a second set of vectors in GSP and described them as images of the two standard basis vectors. With these elements in place, he asked his students to imagine the same linear combination with these different vectors and finally showed them the same linear combination under the new basis. Reflecting on his students, David thought that this GSP investigation seemed to help students connect the linear combinations of the preimage basis to the linear combinations of the image basis "a lot more than they had been."

Embodied and symbolic to formal connection After they "had messed around with *GSP* and the students had asked questions about how things were connected," David made a pedagogical decision to try to connect the embodied *GSP* demonstration and previously presented symbolic thinking. He attempted to make this connection by displaying a transformation visually with GSP and asking the class to "determine the matrix that represented that transformation and verify the images of specific input vectors." David was ecstatic that this intervention had been so successful and useful, as he writes "the *GSP* allowed them to quickly identify an LC of the input basis and calculate the same LC of the output basis and THAT'S what I wanted!" David thought the best part of the *GSP* demonstration was being able to change the basis vectors and immediately see the consequences of that change, particularly when changing the images of the input vectors. Finally, David connected the embodied *GSP* task to the formal properties of an arbitrary basis. David writes in his journal, "So then, I generalized the discussion to talk about any basis and the image of that basis. From this followed the uniqueness of LCs of the basis and its ability to span the domain".

In summary, we saw how a formal goal was set, then when symbolic reasoning was unsuccessful, a shift to the embodied world enabled the instructor to create meaning and used it to connect back to the symbolic and finally to the formal world. Also, this time, he anticipated that students will not be able to make connections and made provision for it.

5 Discussion and conclusions

Given that the research team used Tall's three worlds as a framework to describe the journals, and David characterized his own thought process as outlined in terms of the codes the research team developed, it seems that the model could not have come about without David playing the role of both researcher and participant. During the research meetings, he used the codes he helped to develop as a researcher to describe his experiences teaching as a participant. Furthermore, during discussions with the research team, it was evident that the model represented a notable trend in his teaching experiences. In the context of Tall's worlds, David was himself shifting through the worlds to help his students accomplish the same shift in order to understand the mathematics. He noticed students struggling with symbolic instruction on a formal topic (linear independence), so he created an embodied intervention such as *GSP*. He then utilized this visual tool to help the students move into the embodied world in their mathematical thinking, thereby circumventing the encountered symbolic blockage. Once students developed embodied thinking, David then drew analogies back to the symbolic world to help the students overcome the initial sticking point.

5.1 The creation of a model and Tall's worlds

Designing a model of instruction was one of the outcomes of this study. We found that several of the episodes within the journals formed a pattern that could fit the model of instruction presented in this paper.

David used each of Tall's worlds to serve for specific functions and to deepen his students' mathematical understanding. To begin with, he kept in mind his overarching goals in the formal world, he had set for himself as part of the course, and drew on the symbolic representations he felt would be necessary to approach the goal. He used embodied reasoning to help students make intuitive sense of the symbols they were using, and synthesized symbolic and embodied reasoning together to help refine his students' mathematical understanding. Successful synthesis of symbolic and embodied reasoning appeared to allow him to more explicitly discuss the goal in the formal world (e.g. the invertible matrix theorem) he had set for the course.

The results from our examples revealed that he experienced difficulty in supporting students' formal or symbolic reasoning, hence reflected on these difficulties and experiences, and shifted toward embodied reasoning to strengthen the students' mathematical reasoning. After dealing adequately with the problematic topic, he then shifted back toward attempting to meet his goal in the formal world more thoroughly. According to Tall, (2013, p. 418) "the framework recognizes the need to balance embodied meaning with operational fluency to provide an emotionally positive sense of growth in mathematical thinking, building from practical experiences to theoretical ways of reasoning."

The model exposed and unraveled some of the complexities of teaching linear algebra. We witnessed time and again that the IOLA and GSP tasks were indispensable in highlighting the mathematical aspects of linearly independent and dependent sets of vectors that David set out to achieve. Moreover, the model captured the precise moments in which he decided to move between the worlds.

Although, it is consistent with a few examples from his journal, we view this as a generalization of how such a process might unfold and so we expect this process might be different for other instructors and in other instances and contexts. This study examined one instructor's usage of shifts between the worlds and the possible effects of these shifts, as seen through the eyes of the instructor himself. In particular, this study suggested that for this instructor at least, shifting from the symbolic to embodied world had the potential effect of helping students overcome formal or symbolic difficulties, by providing a more intuitive demonstration of the mathematical concept. Although, we also witnessed the number of challenges that he faced during these shifts.

5.2 Collaborating and including the mathematics instructor in the research process

In line with description of narrative studies, one of the noteworthy features of this study was that the instructor was part of the research team and one of the co-authors of this paper, resulting in an informed analysis of the data. The weekly meetings with the research team likely influenced and potentially helped David refine his own reflections and provided a space to discuss his mathematical ideas that he wanted to develop further. Thus, while we did not see any reflection-in-action due to a lack of classroom data, we were able to examine David's reflections going through a more detailed level of deliberate reflection at these weekly meetings which paralleled the teaching decisions model presented above across a shorter span of time. Moreover, it was beneficial for David to be part of a team who were aware of the research in pedagogy of linear algebra.

Moreover, as David wrote the journals, he implicitly selected aspects of the course that were significant to him, and as such the data from this study naturally enabled us to focus on those aspects of the course that mattered most to the instructor. By the nature of a journal written outside of class, these reflections were not reflections-in-action, but rather primarily reflections-for-action and reflections-on-action. We believe that providing an opportunity and a platform for David to freely reflect on his teaching was fruitful. His journals and reflections enabled him to rethink his ideas and action plans before going to the next class. Furthermore, his journals enabled the research team to get a glimpse inside the mind of a working mathematics instructor.

How might examining one instructors' movements between Tall's worlds contribute to linear algebra instruction? We believe the pedagogical implications of our study are noteworthy. The two episodes described in this study, reminded us of the complexities present in teaching a topic that students often find difficult to understand. Following David's journals closely, we came to appreciate that he had brief windows of opportunity and it was crucial to act quickly and decide on what world to move into. If David did not make those pedagogical decisions to move or did not have the appropriate resources (GSP or his mathematical resources) to move, he had to stay in that world and hope for his students to make sense of the concept. We believe that his moves were carefully calculated and well-served the students. We acknowledge that in general, the know-how and the level of fluency for movements between the worlds relies on paying attention to students' reactions and may be enhanced by more experience in teaching.

5.3 Concluding remarks

Through David's model of instructional decision making, we were able to identify a recurring theme that aligned with and informed shifts between Tall's worlds. Specifically, we identified instances in which David's initial goal was to support students' thinking in the symbolic world. In response to experiences of difficulty, David would shift to the symbolic world to build intuition. This would then be followed by a shift back to the symbolic and formal worlds. These shifts were supported by the instructional resources that David had on hand, such as his meeting with the research team and his experience with developing the curriculum the IOLA project. Typically, David justified his shifts between activities—and thus, Tall's worlds—through his reflections on his classroom experiences, his students' mathematics, and his desired goals for the mathematics of the classroom.

In this study we found that David moved between the worlds when he noticed students are not making connections in the world he was in. His rational was that different representations might help make the connections clear by building on what the student already knows. He had to deal with his frustrations since even some of his carefully thought out interventions still did not work and had to be modified again. The research team came to appreciate a strong connection between the effective instruction and intense reflection and subsequent discussions of that reflection within a team of researchers. Through the use of multiple lenses and a rich data set, we were able to model a small part of David's decision-making process outside of class as he wrestled with the variety of instances inherent in teaching linear algebra. However, there are still questions that remain. While we were able to model David's reflections and pedagogical decisions as they occurred in a potentially more relaxed out-of-classroom environment, we did not have any data to characterize similar decisions that occurred in the heat of the moment inside the classroom. Furthermore, although the results of this study tracked some of David's movements between Tall's worlds, we did not have any data regarding students' reactions to these movements. As we mentioned earlier, this article described one of the studies within the boundaries of the first-named author's research program. As such, more exploration and research on how and why mathematics instructors move between Tall's worlds are in progress.

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