EXTENDING MULTIPLE CHOICE FORMAT TO DOCUMENT STUDENT THINKING

The purpose of this preliminary report is to introduce a new type of assessment instrument to the mathematics education research community and to reflect with our colleagues about the possible affordances and constraints of this instrument. The questions that comprise the instrument consist of a multiple choice (MC) stem followed by a series of options from which students choose explanations (E) that support their multiple choice response. We call this style of question multiple-choice with explanation (MCE). Our decision to use MCE style questions is informed by cutting edge work in physics education research (Wilcox & Pollock, 2013) and introduces an innovative idea for assessing student thinking in mathematics. Our work is part of a larger project in linear algebra (Authors, year); as such, the mathematical context of the assessment instrument is linear algebra. The format of the questions, however, could be used for other subject matter as well.

Key words: Assessment instrument, span, linear independence, student thinking

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Literature Review and Theoretical Perspective

We conducted a two-part literature review, investigating both frameworks for characterizing student understanding in linear algebra, as well as conceptually oriented assessment instruments in undergraduate mathematics and physics. Within linear algebra, we consulted well-known works that had potential to inform our work in characterizing what it means to understand linear algebra, such as Hillel's (2000) modes of description, Sierpinska's (2000) modes of reasoning, Stewart and Thomas's (2009) dual framework utilizing Tall's 3 Worlds and APOS Theory, Authors' (year) interpretations of the matrix equation $A\mathbf{x} = \mathbf{b}$, Authors' (year) concept images through modes of mathematical activity, and Authors' (year) within- and between-concept connections in linear algebra. As we move forward with the assessment we are developing and evaluating it by considering both the structure of the linear algebra concepts themselves and the framing of connections within and between these concepts.

Our review of the literature focusing on conceptually-oriented assessment instruments in undergraduate mathematics and physics revealed five main relevant sources: the Precalculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010), the Force Concept Inventory (Hestenes, Wells, & Swackhamer, 1992), the Calculus Concept Inventory (Epstein, 2013), the multiple choice format of the Quantum Mechanics Assessment Tool (Sadaghiani, Miller, Pollock, & Rehn, 2013), and the multiple choice adaptation of the Colorado Upper-division Electrostatics

(CUE) diagnostic (Wilcox & Pollock, 2013). In particular, Wilcox and Pollock discussed their methods for adapting a valid and reliable free-response assessment instrument into a multiple choice style exam (with question format similar to our MCE format) that preserved the instrument's validity and reliability. We found this work especially useful because of its efforts to bridge the gap between open- and closed-ended questions yet still illuminate students' conceptual understanding.

Methods

We developed the assessment instrument in four phases: (a) reviewing literature and compiling possible questions, (b) developing MCE style questions, (c) piloting the assessment instrument, and (d) analyzing and refining the MCE questions. We began the first phase by exploring all questions used by our team over six separate studies in the previous four years as well as possible questions that existed in the research corpus, consulting various web resources (e.g., http://mathquest.carroll.edu/) and published papers (e.g., Britton & Henderson, 2009; Hamdan, 2005; Rensaa, 2007; Stewart & Thomas, 2009). We focused on questions that specifically addressed the concepts of span and linear independence. The team then iteratively reviewed and trimmed the question list, focusing on questions thought to elicit key aspects of student understanding. We then developed a taxonomy of possible representational systems through which the assessment questions might be asked or across which conceptual understandings might be related.

In the second phase, the research team used this taxonomy and question list to develop thirteen pilot MCE questions. This adaptation focused on developing (E) responses most likely to reflect students' understanding of relationships between concepts and different representations of the same concept. In the third phase, eight of the questions were used in individual interviews (Bernard, 1988) with seven students at two different universities; the other five questions were piloted with four students at one of these universities. We then used four of these MCE questions to conduct a larger pilot implementation with 124 students in five classrooms at three different universities. Student responses were collected, digitally scanned, and coded in a spreadsheet format, the organization of which we describe in the next section.

Results and Discussion

In this section we present three types of analyses that are possible with data from MCE style questions: (1) grid of mathematical relationships, (2) student-focused matrices, and (3) coincidence matrices (and associated Venn diagrams). Type 1 focuses on relationships between the MC and (E) parts of each question, type 2 focuses on patterns in students' responses across questions, and type 3 focuses on a group's responses to a particular question. As this is a preliminary report, we illustrate the beginnings of each analysis type and indicate what we think are interesting points to consider.

Grid of mathematical relationships

Figure 1 illustrates the grid of relationships between the multiple choice (MC) stem of the question and the explanation (E) section. Some questions have 2-4 MC options (See Appendix A), but Figure 1 only includes the correct answer for each. These are listed across the top as column headings. Along the left side, the rows are the possible (E) responses. Within the grid the appropriate column and row intersection is blank if the (E) response was not part of the question in that column. If it was listed as a possible (E) response, then the intersection box indicates the

(E) response number. For example, in Question 1, “Vectors span all of $\mathbb{R}^m$” was (E) response (v). The color of each indicates the relationship of the response to the multiple choice answer. Green indicates that the answer is true and relevant and thus should have been chosen. Red indicates that the response is false and therefore should not have been chosen. Black indicates that the response is true but should not have been chosen because it is not relevant to the problem. For example, “Vectors span all of $\mathbb{R}^m$” was not relevant to whether the set of three vectors in R^2 was linearly dependent, so “v” is in black font for the corresponding cell in Figure 1.

At minimum the grid gives us an indication of what types of relationships we are testing for with these questions. This should allow us to note if there is a relationship that we wished to test for and have not or if there is something that we have tested for more than once. In the latter case, we may choose to eliminate the extra to be efficient or use that opportunity to triangulate information about a student’s responses. We would like to expand our use of the gridding tool to indicate more about individual student thinking. For example, we could indicate whether a student chose certain (E) responses in coordination with certain MC answers to get a snapshot of an individual student’s understanding. One constraint we have with this method is that when a student does not choose a particular (E) response, we cannot be certain whether they did not choose it because they think it is false or because they think it is irrelevant.

	Question				
	1	2		3	4
	Set S is linearly dependent	Matrix as a transformation is onto $\mathbb{R}^2$	Matrix as a transformation is not 1-1	A is invertible when n ≠ 10	Vectors span all of $\mathbb{R}^m$
S includes the *0* vector	i				i
# vectors > # entries	ii				ii
No vectors is a scalar multiple of the others	iii				iii
The only soln to homogeneous eqn. is trivial soln	iv	iii	iii		iv
Vectors span all of R^m (m = # entries in vector)	v				
Matrix made from a subset of S row reduces to I	vi			i, ii, iii	vi
Set S is linearly dependent		i	i	iv, v, vi	
T: $R^m \rightarrow R^n$, $m > n$		ii	ii		
Row-reduced M has a pivot for each row		iv	iv		
The range of T is all of the codomain		v	v		
Set S is linearly independent					v
Two vectors form a linearly independent subset					vii
Two vectors do not lie on the same line					viii
Two vectors lie on the same line					ix

Figure 1. Grid indicating relationships between the MC and (E) parts for each question

Student-focused matrices

We focus on individual student thinking by using a matrix such as that in Figure 2. Across the top are the four question numbers and the (E) response numbers for each. Each row represents one of 29 different students. Under each question column, the student MC response is listed first followed by a “1” in each column of an (E) response that the student chose and a “0” in each column of an (E) response that the student did not choose for that particular question.

Through displaying the data in this way, we examine the variety of responses for an individual student. The data in one row indicates all responses for one student and could be re-gridded into the format of Figure 1 to examine some aspects of that student’s response. In addition, through Figure 2 we can see how a student’s response corresponds to other students’ responses across questions. For example, in Question 1 we see that all students who answered B also chose (i), whereas only about half of the students who incorrectly chose A also chose (i). Because (i) states that the set contains the zero vector, it is likely that those who (incorrectly) answered A (the set is linearly independent) noticed that the zero vector was included in the set but thought it irrelevant to their claim of linear independence. Conversely, only two students who chose B chose (vi), whereas about half who chose A chose (vi). This is sensible because students who thought the set was linearly independent may associate independence with row reducing to the identity matrix.

Another noteworthy aspect is whether students who chose one of the (E) responses chose another related response. For example, in Figure 2, all students who chose B for Question 1 chose the (E) response (i) and half of those also chose (ii), much more than any other (E) choice for Question 1. We consider these types of paired responses in the next section.

Student	Question 1							Question 2						Question 3							Question 4									
		i	ii	iii	iv	v	vi		i	ii	iii	iv	v		i	ii	iii	iv	v	vi		i	ii	iii	iv	v	vi	vii	viii	ix
1	A	0	0	1	0	1	1	A	1	0	0	1	1	B	0	0	0	0	1	0	A	0	0	1	1	0	1	1	1	0
2	A	1	0	1	0	1	0	B	0	1	0	1	0	B	0	1	0	1	0	0	A	0	0	0	1	0	1	0	1	1
6	A	0	0	0	1	1	1	B	0	1	0	1	1	B	0	1	0	0	1	0	C	0	0	0	1	1	1	1	1	0
8	A	1	0	0	0	0	1	B	0	1	0	1	1	B	0	1	0	0	1	0	C	0	0	0	0	1	0	0	1	0
7	A	0	0	0	0	1	1	B	0	1	0	0	0	B	0	1	0	0	0	0	C	0	0	1	0	0	1	1	1	0
3	A	1	1	1	0	1	0	B	0	0	0	1	1	B	0	1	0	1	0	0	C	0	0	1	0	0	0	1	1	0
4	A	1	0	1	0	1	1	B	1	0	0	1	0	B	0	1	0	1	0	0	C	0	0	1	0	1	1	1	1	0
5	A	0	0	1	1	0	1	B	0	1	0	1	1	B	0	1	0	1	0	0	C	0	0	1	1	1	1	1	1	0
9	A	1	0	0	0	1	0	B	0	0	0	1	1	C	1	0	0	0	0	1	C	0	0	0	0	0	0	1	1	0
10	A	1	1	1	0	1	1	C	0	0	0	1	1	B	0	1	0	0	0	0	C	0	0	1	0	1	1	1	1	0
11	A	0	0	1	0	1	1	D	0	1	0	0	0	B	0	1	0	1	0	0	C	0	0	1	1	1	1	0	0	0
12	A	0	0	0	1	0	1	D	1	0	0	1	0	B	0	1	0	1	0	0	D	0	0	1	0	0	0	0	1	0
13	A	0	0	1	1	0	1	D	1	1	0	1	1	NR	0	1	0	1	0	0	C	0	0	0	1	1	1	1	1	0
14	B	1	0	1	0	1	1	A	0	1	0	1	1	B	0	1	0	0	0	1	C	0	0	1	0	0	0	1	1	0
15	B	1	1	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	0	0	B	0	0	0	0	0	0	1	1	0
16	B	1	0	0	0	1	1	B	0	0	0	1	0	B	0	1	0	0	1	0	C	1	0	0	0	0	1	1	1	1
18	B	1	0	0	0	0	0	B	0	1	0	1	1	B	0	1	0	1	0	0	C	0	0	0	0	0	1	1	1	0
19	B	1	1	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	1	0	C	0	0	0	0	0	1	1	1	0
21	B	1	0	0	0	0	0	B	0	0	0	1	1	B	0	1	0	0	0	0	C	0	0	0	0	0	0	1	1	0
22	B	1	0	0	0	1	0	B	0	0	0	1	0	B	0	1	0	0	1	0	C	1	1	0	0	0	0	0	0	1
17	B	1	0	1	1	0	0	B	0	0	1	1	0	B	0	0	1	0	0	0	C	0	0	1	0	1	0	0	0	0
20	B	1	1	1	0	0	0	B	0	1	0	0	1	B	0	1	0	0	0	0	C	0	0	1	0	0	0	0	0	0
23	B	1	1	0	0	0	0	B	0	1	0	1	1	B	0	1	0	1	0	0	D	0	1	0	0	0	1	1	1	1
24	B	1	1	0	0	0	0	B	1	1	0	0	0	B	0	1	0	1	0	0	D	0	0	1	0	0	1	1	1	0
25	B	1	1	1	0	1	0	B	1	1	0	1	0	C	0	0	0	0	0	1	D	1	1	0	0	0	0	1	1	1
26	B	1	1	0	0	0	0	B	1	0	0	1	0	C	1	0	0	0	0	0	NR	0	0	1	0	0	1	1	1	1
27	B	1	0	0	0	1	0	D	0	1	0	1	0	A	0	0	1	1	0	0	C	0	0	1	0	0	1	0	1	0
28	B	1	0	1	0	0	0	D	1	0	0	1	1	B	0	1	0	0	0	0	A	1	1	1	0	0	1	1	1	1
29	B	1	1	0	0	0	0	D	1	1	0	0	0	B	0	1	0	1	0	0	C	0	0	0	0	0	1	1	0	0

Figure 2. Student responses Fall 2013

Coincidence matrices

To check for pairs of student (E) responses we have constructed coincidence matrices. For instance, Figure 3 shows the participants' selection of (E) responses for Question 1, sorting the (E) responses by MC response. In these matrices, each entry gives the number of participants who chose both the (E) response in a given row and also the (E) response in a given column. The diagonal entries show the number of participants who chose the particular (E) response. For instance, in the matrix on the left, the entry in Row i, Column i shows that only 19 of the 49 students who incorrectly chose MC response A (that the set of vectors $\left\{\begin{bmatrix}4\\5\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is linearly independent) also correctly chose (E) response (i) (that the set contains the zero vector). As discussed above these students likely thought that the zero vector was irrelevant to determining that the set is linearly independent. On the other hand, 73 of the 75 participants who correctly chose MC response B (the aforementioned set is linearly dependent) also correctly chose (E) response (i). In addition, 54 of the 75 participants who chose MC response B also chose (E) response (ii) (that the set is 3 vectors in $\mathbb{R}^2$). All but one of these participants also chose (E) response (i) (note the 53 in the Row i, Column ii cell).

Another example of a coincidence matrix shows participants who have correctly answered the MC part of Question 4. We see that these students have widely varying responses to the (E) part. The three correct (E) responses are (vi), (vii), and (viii). The students who chose these responses and pairs of these responses are highlighted in green in the right of Figure 4. The box in the lower right corner shows the total number of participants (32) who correctly responded to both the MC and (E) parts of this question. This example shows one of the limitations of the coincidence matrix organization of the data. Specifically, the coincidence matrix only provides pair-wise counts of coincidental responses, whereas it may be advantageous to consider whether more than two (E) responses are in common for a given student. This matrix can be supplemented with a Venn Diagram showing how the responses intersect. On the left of Figure 4 is a Venn diagram that breaks down the 93 students who answered C into the 7 who chose none of the correct (E) responses, the 32 who chose all three of the correct (E) responses and other categories of pairs or singular responses from the three (E) correct responses.

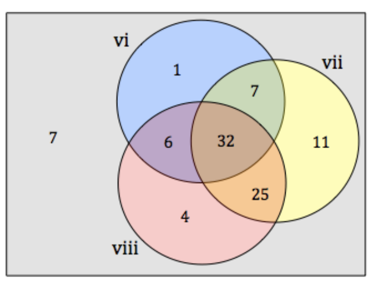
MC Selection: A

49	i	ii	iii	iv	v	vi
i	19	7	11	6	12	7
ii		10	3	5	5	4
iii			29	10	15	12
iv				24	10	9
v					24	11
vi						21

MC Selection: B

75	i	ii	iii	iv	v	vi
i	73	53	10	8	20	6
ii		54	6	6	14	4
iii			11	4	5	3
iv				9	4	2
v					20	6
vi						6

Figure 3. Coincidence matrices for Question 1, separated by MC response



C

93	i	ii	iii	iv	v	vi	vii	viii	ix	
i	15	2	6	3	4	7	10	10	4	
ii		10	6	1	1	5	8	7	4	
iii			37	7	16	20	26	28	4	
iv				12	6	9	9	9	3	
v					24	12	14	16	2	
vi						46	39	38	5	
vii							75	57	10	
viii								67	4	
ix									12	32

Figure 4. Venn diagram and coincidence matrix for Question 4, Response C

We would like the audience to think with us about the affordances and constraints of the MCE style question format, as well as these three ways of organizing the data for analysis. In particular, what types of student thinking can be measured with the MCE style questions? How can we best leverage the different data organization methods to get at student thinking? In what ways do the various analyses help us study individuals versus classrooms?

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Appendix A: MCE questions as implemented in 2013 and 2014

1) The set of vectors $\left\{\begin{bmatrix}4\\5\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ is
   A) Linearly independent
   B) Linearly dependent
   Because ... (Select ALL that support you choice)
   i) the set includes the *0* vector.
   ii) the set has 3 vectors in $\mathbb{R}^2$.
   iii) none of the vectors are multiples of each other.
   iv) the only solution to $c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} + c_3\begin{bmatrix}0\\0\end{bmatrix} = 0$ is the trivial solution.
   v) the vectors span all of $\mathbb{R}^2$.
   vi) $\begin{bmatrix}4 & 2\\5 & 3\end{bmatrix}$ is row equivalent to the identity matrix

2) The transformation defined by $T(x) = Mx$, where the matrix is $M = \begin{bmatrix}2 & 4 & 6\\1 & 3 & 3\end{bmatrix}$
   A) One-to-one
   B) Onto
   C) Both
   D) Neither
   Because ... (Select ALL that support you choice)
   i) the columns of the matrix are linearly dependent.
   ii) the matrix maps $\mathbb{R}^3$ to $\mathbb{R}^2$.
   iii) the only solution to $c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}4\\3\end{bmatrix} + c_3\begin{bmatrix}6\\3\end{bmatrix} = 0$ is the trivial solution.
   iv) the row-reduced echelon form of M has two pivots.
   v) the range of this transformation is all of $\mathbb{R}^2$

3) Let $A = \begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$. For what value(s) of n is the matrix A invertible?
   A) $n = 10$ B) $n \neq 10$ C) $n \in \mathbb{R}$
   Because ... (Select ALL that support you choice)
   i) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for all $n \in \mathbb{R}$, row-reduces to the identity matrix.
   ii) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for all $n \neq 10$, row-reduces to the identity matrix.
   iii) $\begin{bmatrix}1 & -2 & 5\\2 & -4 & n\\0 & 1 & 1\end{bmatrix}$, for $n = 10$, row-reduces to the identity matrix.
   iv) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n = 10$.
   v) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n \neq 10$.
   vi) the set of vectors $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}-2\\-4\\1\end{bmatrix}, \begin{bmatrix}5\\n\\1\end{bmatrix}\right\}$ is linearly dependent for $n \in \mathbb{R}$.

4) The set of vectors $\left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ spans
   A) a point in $\mathbb{R}^2$
   B) a line in $\mathbb{R}^2$
   C) all of $\mathbb{R}^2$
   D) a plane in $\mathbb{R}^3$
   E) all of $\mathbb{R}^3$
   Because ... (Select ALL that support you choice)
   i) the set includes $0 \in \mathbb{R}^2$.
   ii) the set has 3 vectors.
   iii) none of the vectors are multiples of each other.
   iv) the only solution to $c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\5\end{bmatrix} + c_3\begin{bmatrix}0\\0\end{bmatrix} = 0$ is the trivial solution.
   v) the set of vectors is linearly independent.
   vi) $\begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$ is row equivalent to the identity matrix.
   vii) two of the vectors in the set are linearly independent.
   viii) the two vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\5\end{bmatrix}$ do not lie on the same line.
   ix) the two vectors $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$ lie on the same line.